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ROUGH PATHS IN IDEALIZED FINANCIAL MARKETS

V. Vovk

Department of Computer Science, Royal Holloway, University of London, Egham, Surrey TW20 0EX, UK
(e-mail: vovk@cs.rhul.ac.uk)

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Abstract. This paper considers possible price paths of a financial security in an idealized market. Its main result is that the variation index of typical price paths is at most 2; in this sense, typical price paths are not rougher than typical paths of Brownian motion. We do not make any stochastic assumptions and only assume that the price path is right-continuous. The qualification "typical" means that there is a trading strategy (constructed explicitly in the proof) that risks only one monetary unit but brings infinite capital when the variation index of the realized price path exceeds 2. The paper also reviews some known results for continuous price paths.

Keywords: game-theoretic probability, incomplete markets, continuous time, discontinuous price paths, variation index

1 INTRODUCTION

"Rough paths" are functions with infinite total variation, and their roughness is usually measured using the notion of p-variation ([12], p. 102). Rough paths are ubiquitous in the theory of stochastic processes, but in recent years they have been actively studied in non-probabilistic settings as well (see, e.g., [6] and [10]). This paper is a contribution to this area of research, studying price paths of financial securities in idealized markets. It comes from the tradition of "game-theoretic probability" (an approach to probability going back to von Mises and Ville). No probabilistic assumptions are made about the evolution of security prices (a non-stochastic notion of probability can be defined, but this step is optional). The early work on price paths in game-theoretic probability relied on using non-standard analysis (as in [15]); this paper follows Takeuchi *et al.*'s recent paper [18] in avoiding non-standard analysis.

We will consider the price path of one financial security over a finite time interval $[0, T]$. Our key assumption is that the market in our security is efficient, in the following weak sense: a prespecified trading strategy risking only 1 monetary unit will not bring infinite capital at time T. This assumption is not required for our mathematical results, but is useful in their interpretation and justifies our terminology: we say that a property holds for typical price paths if there is a trading strategy risking only 1 monetary unit that brings infinite capital at time T whenever the property fails. Our other assumption is that the interest rate over the time interval $[0, T]$ is 0; this assumption is easy to relax and is made only for simplicity.

Let $\omega : [0, T] \to [0, \infty)$ be the price path of our financial security; in this paper we always assume that it is positive (meaning $\omega \geq 0$; this assumption is usually satisfied in real markets). Section 2 discusses the case where ω is known to be right-continuous. In this case we can only prove that the p-variation of a typical ω is finite when $p > 2$. In Section 3 we consider the case where ω is known to be continuous. In this case our understanding is deeper and we describe briefly some of the much stronger results obtained in [21]. A typical result is that the p-variation of a non-constant typical ω is finite when $p > 2$ and infinite when $p \leq 2$; in particular, the variation index of a typical ω is either 0 or 2. In the last section, Section 4, we consider markets where borrowing (both borrowing cash and borrowing securities) is prohibited; for such markets, the assumption that ω is continuous loses much of its power.

Our approach to rough paths is somewhat different from the standard one, introduced by Lyons [10]. Lyons's theory can deal directly only with the rough paths ω satisfying $\mathrm{vi}(\omega) < 2$ (by means of Youngs' theory, which is described in, e.g., [6], Section 2.2). In order to treat rough paths satisfying $\mathrm{vi}(\omega) \in [n, n+1)$, where $n = 2, 3, \ldots$, we need to postulate the values of the iterated integrals $X^i_{s,t} := \int_{s<u_1<\cdots<u_i<t} \mathrm{d}\omega(u_1) \cdots \mathrm{d}\omega(u_i)$ for $i = 2, \ldots, n$ and $0 \le s < t \le T$ (satisfying "Chen's consistency condition"). It is not clear how to avoid making an arbitrary choice here. Our main result (Theorem 1) says that only the case $n = 2$ is relevant for our idealized markets, and in this case Lyons's theory is much simpler than in general; its application in the context of this paper becomes much more feasible, and would be an interesting direction of further research.

For further discussion of connections with the standard theory of mathematical finance, including the First and Second Fundamental Theorems of Asset Pricing and various versions of the no-arbitrage condition, see [21], Sections 1 and 12.

This paper is based on my talks with the same title at the Tenth International Vilnius Conference on Probability and Mathematical Statistics (section "Random Processes", session "Rough Paths", 29 June 2010) and the Third Workshop on Game-theoretic Probability and Related Topics (21 June 2010).

The words "positive" and "increasing" are always understood in the wide sense of "$\ge$"; the adverb "strictly" will be added when needed. Our notation for logarithms is $\ln$ (natural) and $\log$ (binary).

2 VOLATILITY OF RIGHT-CONTINUOUS PRICE PATHS

Let Ω be the set of all positive right-continuous functions $\omega : [0, T] \to [0, \infty)$; we will call Ω our *sample space*. For each $t \in [0, T]$, $\mathcal{F}^\circ_t$ is defined to be the smallest σ-algebra on Ω that makes all functions $\omega \mapsto \omega(s)$, $s \in [0, t]$, measurable; $\mathcal{F}_t$ is defined to be the universal completion of $\mathcal{F}^\circ_t$. A *process* S is a family of functions $S_t : \Omega \to [-\infty, \infty]$, $t \in [0, T]$, each S_t being $\mathcal{F}_t$-measurable (we drop the adjective "adapted"). An *event* is an element of the σ-algebra $\mathcal{F}_T$. Stopping times $\tau : \Omega \to [0, T] \cup \{\infty\}$ w.r. to the filtration $(\mathcal{F}_t)$ and the corresponding σ-algebras $\mathcal{F}_\tau$ are defined as usual; $\omega(\tau(\omega))$ and $S_{\tau(\omega)}(\omega)$ will be simplified to $\omega(\tau)$ and $S_\tau(\omega)$, respectively (occasionally, the argument ω will be omitted in other cases as well).

The class of allowed trading strategies is defined in two steps. A *simple trading strategy* G consists of the following components: $c \in \mathbb{R}$ (the *initial capital*); an increasing sequence of stopping times $\tau_1 \le \tau_2 \le \cdots$; and, for each $n = 1, 2, \ldots$, a bounded $\mathcal{F}_{\tau_n}$-measurable function h_n. It is required that, for any $\omega \in \Omega$, only finitely many of $\tau_n(\omega)$ should be finite. (Including the initial capital in the trading strategy is a standard convention in mathematical finance.) To such G corresponds the *simple capital process*

$$\mathcal{K}^G_t(\omega) := c + \sum_{n=1}^{\infty} h_n(\omega) \bigl(\omega(\tau_{n+1} \wedge t) - \omega(\tau_n \wedge t)\bigr), \quad t \in [0, T] \tag{2.1}$$

(with the zero terms in the sum ignored); the value $h_n(\omega)$ will be called the *position* taken at time τ_n, and $\mathcal{K}^G_t(\omega)$ will sometimes be referred to as the capital process of G.

A *positive capital process* is any process S that can be represented in the form

$$S_t(\omega) := \sum_{m=1}^{\infty} \mathcal{K}^{G_m}_t(\omega), \tag{2.2}$$

where the simple capital processes $\mathcal{K}^{G_m}_t(\omega)$ are required to be positive, for all $t \in [0, T]$ and $\omega \in \Omega$, and the positive series $\sum_{m=1}^{\infty} c_m$ is required to converge, where c_m is the initial capital of G_m. The sum (2.2) is always positive but allowed to take value ∞. Since $\mathcal{K}^{G_m}_0(\omega) = c_m$ does not depend on ω, $S_0(\omega)$ also does not depend on ω and will sometimes be abbreviated to S_0. In our discussions we will sometimes refer to the sequence $(G_m)_{m=1}^{\infty}$ as a *trading strategy risking* $\sum_m c_m$ and refer to (2.2) as the *capital process* of this strategy.

Remark 1. The intuition behind the definition of positive capital processes is that the initial capital is split into infinitely many accounts and the trader runs a separate simple trading strategy on each of these accounts. Our definition of simple trading strategies only involves the position taken in security, not the cash position. The cash position is determined uniquely from the condition that the strategy should be self-financing (see Section 4 for further details), and in many cases there is no need to mention it explicitly. For the explicit connection between our notion of a simple trading strategy and the standard definition of a self-financing trading strategy (specifying explicitly the cash position, as in, e.g., [16], Section VII.1a), see [21], Subsection 2.1.

We say that a set $E \subseteq \Omega$ is *null* if there is a positive capital process S such that $S_0 = 1$ and $S_T(\omega) = \infty$ for all $\omega \in E$. A property of $\omega \in \Omega$ will be said to hold *almost surely* (a.s.), or *for typical* ω, if the set of ω where it fails is null. Intuitively, we expect such a property to be satisfied in a market that is efficient at least to some degree.

For each $p \in (0, \infty)$, the *p-variation* $\mathrm{v}_p(f)$ of a function $f : [0, T] \to \mathbb{R}$ is defined as

$$\mathrm{v}_p(f) := \sup_{\kappa} \sum_{i=1}^{n} |f(t_i) - f(t_{i-1})|^p , \tag{2.3}$$

where n ranges over all strictly positive integers and κ over all *partitions* $0 = t_0 \leq t_1 \leq \cdots \leq t_n = T$ of the interval $[0, T]$. The *total variation* of a function is the same thing as its 1-variation. It is obvious that, when f is bounded, there exists a unique number $\mathrm{vi}(f) \in [0, \infty]$, called the *variation index* of f, such that $\mathrm{v}_p(f)$ is finite when $p > \mathrm{vi}(f)$ and infinite when $p < \mathrm{vi}(f)$. It is easy to see that $\mathrm{vi}(f) \notin (0, 1)$ when f is continuous, but in general $\mathrm{vi}(f)$ can take any values in $[0, \infty]$.

Theorem 1. *For typical* $\omega \in \Omega$,

$$\mathrm{vi}(\omega) \leq 2. \tag{2.4}$$

In the case of semimartingales, the property (2.4) was established by Lepingle ([8], Theorem 1(a)). Intuitively, Theorem 1 says that price paths cannot be too rough. In fact, this theorem can be strengthened to say that there is a trading strategy risking at most 1 monetary unit whose capital process is ∞ at any time t such that the variation index of ω over $[0, t]$ is greater than 2. (This remark is also applicable to all other results of this kind in this paper.) Theorem 1 will be proved using Stricker's [17] method (which is an extension of Bruneau's [1] method from continuous to càdlàg functions).

Let $\mathrm{M}_a^b(f)$ (resp. $\mathrm{D}_a^b(f)$) be the number of upcrossings (resp. downcrossings) of an open interval (a, b) by a function $f : [0, T] \to \mathbb{R}$ during the time interval $[0, T]$. For each $h > 0$ set

$$\mathrm{M}(f, h) := \sum_{k \in \mathbb{Z}} \mathrm{M}_{kh}^{(k+1)h}(f), \quad \mathrm{D}(f, h) := \sum_{k \in \mathbb{Z}} \mathrm{D}_{kh}^{(k+1)h}(f).$$

The key ingredient of the proof of Theorem 1 is the following game-theoretic version of Doob's upcrossings inequality:

Lemma 1. *Let $0 \leq a < b$ be real numbers. There exists a positive simple capital process S that starts from $S_0 = a$ and satisfies, for all $\omega \in \Omega$,*

$$S_T(\omega) \geq (b - a)\, \mathrm{M}_a^b(\omega). \tag{2.5}$$

Proof. The following standard argument will be easy to formalize. A simple trading strategy G leading to S can be defined as follows. The initial capital is a. At first G takes position 0. When ω first hits $[0, a]$, G takes position 1 until ω hits $[b, \infty)$, at which point G takes position 0; after ω hits $[0, a]$, G maintains position 1 until ω hits $[b, \infty)$, at which point G takes position 0; etc. Since ω is positive, S will also be positive.

Formally, we define $\tau_1 := \inf\{t \mid \omega(t) \in [0,a]\}$ and, for $n = 2, 3, \ldots$,

$$\tau_n := \inf\{t \mid t > \tau_{n-1} \mathbin{\&} \omega(t) \in I_n\},$$

where $I_n := [b, \infty)$ for even n and $I_n := [0, a]$ for odd n. (As usual, the expression $\inf \emptyset$ is interpreted as ∞.) Since ω is a right-continuous function and $[0,a]$ and $[b,\infty)$ are closed sets, the infima in the definitions of $\tau_1, \tau_2, \ldots$ are attained. Therefore, $\omega(\tau_1) \le a$, $\omega(\tau_2) \ge b$, $\omega(\tau_3) \le a$, $\omega(\tau_4) \ge b$, and so on. The positions taken by G at the times $\tau_1, \tau_2, \ldots$ are $h_1 := 1$, $h_2 := 0$, $h_3 := 1$, $h_4 := 0$, etc., and the initial capital is a. Let n be the largest integer such that $\tau_n \le T$ (with $n := 0$ when $\tau_1 = \infty$). Now we obtain from (2.1): if n is even,

$$\begin{aligned} S_T(\omega) &= \mathcal{K}_T^G(\omega) \\ &= a + (\omega(\tau_2) - \omega(\tau_1)) + (\omega(\tau_4) - \omega(\tau_3)) + \cdots + (\omega(\tau_n) - \omega(\tau_{n-1})) \\ &\ge a + (b-a)\,\mathrm{M}_a^b(\omega), \end{aligned}$$

and if n is odd,

$$\begin{aligned} S_T(\omega) &= \mathcal{K}_T^G(\omega) \\ &= a + (\omega(\tau_2) - \omega(\tau_1)) + (\omega(\tau_4) - \omega(\tau_3)) + \cdots + (\omega(\tau_{n-1}) - \omega(\tau_{n-2})) + (\omega(t) - \omega(\tau_n)) \\ &\ge a + (b-a)\,\mathrm{M}_a^b(\omega) + (\omega(t) - \omega(\tau_n)) \\ &\ge a + (b-a)\,\mathrm{M}_a^b(\omega) + (0 - a) = (b-a)\,\mathrm{M}_a^b(\omega); \end{aligned}$$

in both cases, (2.5) holds. In particular, $S_T(\omega)$ is positive; the same argument applied to $t \in [0,T]$ in place of T shows that $S_t(\omega)$ is positive for all $t \in [0,T]$.

It remains to check that each τ_n is a stopping time; we will do so using induction in n. Let $t \in [0,T]$. Since ω is right-continuous and $[0,a]$ is closed, the set $\{\tau_1 \le t\}$ is the projection onto Ω of the set $A := \{(s,\omega) \in [0,t] \times \Omega \mid \omega(s) \in [0,a]\}$ (cf. [4], IV.51(c)). Since $A \in \mathcal{B}_t \times \mathcal{F}_t^\circ$, where $\mathcal{B}_t$ is the Borel σ-algebra on $[0,t]$ and $\mathcal{B}_t \times \mathcal{F}_t^\circ$ is the product σ-algebra, the projection $\{\tau_1 \le t\}$ is an $\mathcal{F}_t^\circ$-analytic set (according to [4], Theorem III.13(3)). Therefore, $\{\tau_1 \le t\} \in \mathcal{F}_t$ (according to [4], Theorem III.33). We can see that τ_1 is a stopping time.

Now let $n \in \{2, 3, \ldots\}$ and suppose that τ_{n-1} is a stopping time. Let $t \in [0,T]$. Since ω is right-continuous and I_n is closed, the set $\{\tau_n \le t\}$ is the projection onto Ω of the set $A := \{(s,\omega) \in [0,t] \times \Omega \mid s > \tau_{n-1} \mathbin{\&} \omega(s) \in I_n\}$. Since $A \in \mathcal{B}_t \times \mathcal{F}_t^\circ$, the same argument as in the previous paragraph shows that $\{\tau_n \le t\} \in \mathcal{F}_t$; therefore, τ_n is a stopping time.

Finally, let us check carefully that the set $\{\tau_n \le t\}$ is indeed the projection onto Ω of $A := \{(s,\omega) \in [0,t] \times \Omega \mid s > \tau_{n-1} \mathbin{\&} \omega(s) \in I_n\}$, assuming $n > 1$ (the corresponding assertion for $n = 1$ is even easier). One direction is trivial: $s \in [0,t]$, $s > \tau_{n-1}$, and $\omega(s) \in I_n$ immediately implies $\tau_n \le t$. In the opposite direction, suppose $\tau_n(\omega) \le t$. There is $s \in [0,t]$ and a sequence $t_1 \ge t_2 \ge \cdots$ such that $\lim_{i\to\infty} t_i = s$ and, for all i, $t_i > \tau_{n-1}(\omega)$ and $\omega(t_i) \in I_n$. Since ω is right-continuous and I_n is closed, $\omega(s) = \lim_{i\to\infty} \omega(t_i) \in I_n$. We cannot have $s = \tau_{n-1}$ since $\omega(s) \in I_n$ and $\omega(\tau_{n-1}) \notin I_n$. $\square$

In fact, in Proposition 1 below we will prove a stronger version of Theorem 1. But to state the stronger version we will need a generalization of the definition (2.3). Let $\phi : [0,\infty) \to [0,\infty)$. For $f : [0,T] \to \mathbb{R}$, we set

$$\mathrm{v}_\phi(f) := \sup_\kappa \sum_{i=1}^n \phi\left(|f(t_i) - f(t_{i-1})|\right),$$

where κ ranges over all partitions $0 = t_0 \le t_1 \le \cdots \le t_n = T$ of $[0,T]$.

Proposition 1. *Suppose* $\phi : (0,\infty) \to (0,\infty)$ *satisfies*

$$\sup_{0<t\le s\le 2t} \frac{\phi(s)}{\phi(t)} < \infty \quad \textit{and} \quad \sum_{j=0}^{\infty} 2^{2j}\phi\left(2^{-j}\right) < \infty. \tag{2.6}$$

Then $\mathrm{v}_\phi(\omega) < \infty$ *a.s., where* $\phi(0)$ *is set to* 0.

Informally, the first condition in (2.6) says that ϕ should never increase too fast, and the second condition says that $\phi(u)$ should approach 0 somewhat faster than u^2 as $u \to 0$. To obtain Theorem 1, set $\phi(u) := u^p$, where $p > 2$ is rational, and notice that the union of countably many null events is always null. Another simple example of a function ϕ satisfying (2.6) is $\phi(u) := (u/\log^* u)^2$, where $\log^* u := 1 \vee |\log u|$. A better example is $\phi(u) := u^2/(\log^* u \log^* \log^* u \cdots)$ (the product is finite if we ignore the factors equal to 1); for a proof of (2.6) for this function, see [9], Appendixes B and C (in this example, it is essential that $\log$ is binary rather than natural logarithm). However, even for the last choice of ϕ, the inequality $\mathrm{v}_\phi(\omega) < \infty$ a.s. is still much weaker than the inequality $\mathrm{v}_\psi(\omega) < \infty$ a.s., with ψ defined by (3.1), which we can prove assuming ω continuous (see Proposition 4 below).

Proof of Proposition 1. Set $w(j) := 2^{2j}\phi(2^{-j})$, $j = 0, 1, \ldots$; by (2.6), $\sum_{j=0}^{\infty} w(j) < \infty$. Without loss of generality we will assume that $\sum_{j=0}^{\infty} w(j) = 1$.

Let $0 = t_0 \le t_1 \le \cdots \le t_n = T$ be a partition of the interval $[0, T]$; without loss of generality we replace all "$\le$" by "$<$". Fix $\omega \in \Omega$; at first we will be mostly interested in the case where $\sup_{t\in[0,T]} \omega(t) \le 2^L$ for a given positive integer L. Split $\sum_{i=1}^{n} \phi\left(|\omega(t_i) - \omega(t_{i-1})|\right)$ into two parts:

$$\sum_{i=1}^{n} \phi\left(|\omega(t_i) - \omega(t_{i-1})|\right) = \sum_{i\in I_+} \phi\left(\omega(t_i) - \omega(t_{i-1})\right) + \sum_{i\in I_-} \phi\left(\omega(t_{i-1}) - \omega(t_i)\right),$$

where

$$I_+ := \{i \mid \omega(t_i) - \omega(t_{i-1}) > 0\},$$
$$I_- := \{i \mid \omega(t_i) - \omega(t_{i-1}) < 0\}.$$

By Lemma 1, for each $j = 0, 1, \ldots$ and each $k \in \{0, \ldots, 2^{L+j} - 1\}$ there exists a positive simple capital process $S^{j,k}$ that starts from $k2^{-j}$ and satisfies

$$S_T^{j,k}(\omega) \ge 2^{-j}\, \mathrm{M}_{k2^{-j}}^{(k+1)2^{-j}}(\omega).$$

Summing $2^{-L-j}S^{j,k}$ over $k = 0, \ldots, 2^{L+j} - 1$, we obtain a positive capital process S^j such that

$$S_0^j = \sum_{k=0}^{2^{L+j}-1} k2^{-L-2j} \le 2^{L-1},$$
$$S_T^j(\omega) \ge 2^{-L-2j}\, \mathrm{M}(\omega, 2^{-j}) \text{ when } \sup\omega \le 2^L.$$

For each $i \in I_+$, let $j(i)$ be the smallest positive integer j satisfying

$$\exists k \in \{0, 1, 2, \ldots\} : \omega(t_{i-1}) \le k2^{-j} \le (k+1)2^{-j} \le \omega(t_i). \tag{2.7}$$

Summing $w(j)S^j$ over $j = 0, 1, \ldots$, we obtain a positive capital process S such that $S_0 \le 2^{L-1}$ and, when $\sup \omega \le 2^L$,

$$S_T(\omega) \ge \sum_{j=0}^{\infty} w(j) 2^{-L-2j} \, \mathrm{M}(\omega, 2^{-j}) \ge \sum_{i \in I_+} w(j(i)) 2^{-L-2j(i)} \tag{2.8}$$

$$= 2^{-L} \sum_{i \in I_+} \phi\left(2^{-j(i)}\right) \ge \delta \sum_{i \in I_+} \phi\left(\omega(t_i) - \omega(t_{i-1})\right), \tag{2.9}$$

where $\delta > 0$ depends only on L and the supremum in (2.6). The second inequality in (2.8) follows from the fact that to each $i \in I_+$ corresponds an upcrossing of an interval of the form $(k2^{-j(i)}, (k+1)2^{-j(i)})$.

An inequality analogous to the inequality between the second and the last terms of the chain (2.8)–(2.9) can be proved for downcrossings instead of upcrossings, I_- instead of I_+, and $\omega(t_{i-1})$ and $\omega(t_i)$ swapped around. Using this inequality (in the third "$\ge$" below) gives, when $\sup \omega \le 2^L$,

$$S_T(\omega) \ge \sum_{j=0}^{\infty} w(j) 2^{-L-2j} \, \mathrm{M}(\omega, 2^{-j}) \ge \sum_{j=0}^{\infty} w(j) 2^{-L-2j} \left(\mathrm{D}(\omega, 2^{-j}) - 2^{L+j}\right)$$
$$\ge \delta \sum_{i \in I_-} \phi\left(\omega(t_{i-1}) - \omega(t_i)\right) - \sum_{j=0}^{\infty} w(j) 2^{-j} \ge \delta \sum_{i \in I_-} \phi\left(\omega(t_{i-1}) - \omega(t_i)\right) - 1.$$

Averaging the two lower bounds for $S_T(\omega)$, we obtain, when $\sup \omega \le 2^L$,

$$S_T(\omega) \ge \frac{\delta}{2} \sum_{i=1}^{n} \phi\left(|\omega(t_i) - \omega(t_{i-1})|\right) - \frac{1}{2}.$$

Taking supremum over all partitions gives

$$\sup \omega \le 2^L \Longrightarrow S_T(\omega) \ge \frac{\delta}{2} \, \mathrm{v}_\phi(\omega) - \frac{1}{2}. \tag{2.10}$$

We can see that the event that $\sup \omega \le 2^L$ and $\mathrm{v}_\phi(\omega) = \infty$ is null. Since the union of countably many null events is always null, the event that $\mathrm{v}_\phi(\omega) = \infty$ is also null. □

As a simple corollary of Theorem 1, we state the following version of Theorem VI.3(2) in [5].

Corollary 1. *Almost surely, the price path $\omega \in \Omega$ is càdlàg.*

Proof. The corollary follows from the fact that any function with finite p-variation (for any $p > 0$) is regulated: see, e.g., [6], Proposition 3.33. □

The case of right-continuous price paths considered in this section is very different from the case of continuous price paths that we take up in the following section. Proposition 3 will show that, in the latter case, $\mathrm{vi}(\omega) \in \{0, 2\}$ a.s. In the former case, no right-continuous price path that is bounded away from zero and has finite total variation can belong to a null event, as the following proposition will show.

The *upper probability* of a set $E \subseteq \Omega$ is defined as

$$\overline{\mathbb{P}}(E) := \inf\left\{S_0 \;\middle|\; \forall \omega \in \Omega : S_T(\omega) \ge \mathbb{I}_E(\omega)\right\}, \tag{2.11}$$

where S ranges over the positive capital processes and $\mathbb{I}_E$ stands for the indicator of E. In this section we will be interested only in one-element sets E. We write $\mathrm{v}(f)$ meaning $\mathrm{v}_1(f)$.

Proposition 2. *For any $\omega \in \Omega$,*

$$\overline{\mathbb{P}}(\{\omega\}) = \sqrt{\frac{\omega(0)}{\omega(T)} e^{-\mathrm{v}(\ln \omega)}}. \tag{2.12}$$

Proof. Fix $\omega \in \Omega$. Let S be any positive capital process. Represent it in the form (2.2). It suffices to prove that none of the component strategies G_m can increase its initial capital c_m by more than a factor of

$$\sqrt{\frac{\omega(T)}{\omega(0)} e^{\mathrm{v}(\ln \omega)}}$$

and that this factor itself is attainable, at least in the limit. Fix an m and let $c = c_m$, $\tau_1, \tau_2, \ldots$, and $h_1, h_2, \ldots$ be the component initial capital, stopping times, and positions of G_m. It is clear that all h_n must be positive in order for $\mathcal{K} := \mathcal{K}^{G_m}$ to be positive: upward price movements are unbounded. Downward price movements right after τ_n can be as large as $\omega(\tau_n)$, which implies that

$$0 \le h_n \le \mathcal{K}_{\tau_n} / \omega(\tau_n) \tag{2.13}$$

(this condition will be further discussed and justified in Section 4). This gives, according to (2.1),

$$\mathcal{K}_{\tau_{n+1}} = \mathcal{K}_{\tau_n} + h_n \left(\omega(\tau_{n+1}) - \omega(\tau_n)\right) \le \left(1 \vee \frac{\omega(\tau_{n+1})}{\omega(\tau_n)}\right) \mathcal{K}_{\tau_n}.$$

The last "$\le$" becomes "$=$" when

$$h_n := \begin{cases} \mathcal{K}_{\tau_n} / \omega(\tau_n) & \text{if } \omega(\tau_{n+1}) > \omega(\tau_n) \\ 0 & \text{otherwise.} \end{cases}$$

We can see that no positive simple capital process increases its initial capital by more than a factor of $e^{\mathrm{v}^+(\ln \omega)}$, where $\mathrm{v}^+(f)$ is defined by the following modification of (2.3):

$$\mathrm{v}^+(f) := \sup_{\kappa} \sum_{i=1}^{n} \left(f(t_i) - f(t_{i-1})\right)^+ ;$$

as usual, u^+ and u^- are defined to be $0 \vee u$ and $0 \vee (-u)$, respectively. On the other hand, for each $\epsilon > 0$, there is a positive simple capital process that increases its initial capital by a factor of at least $(1-\epsilon) e^{\mathrm{v}^+(\ln \omega)}$. We can see that

$$\overline{\mathbb{P}}(\{\omega\}) = e^{-\mathrm{v}^+(\ln \omega)}. \tag{2.14}$$

If we define

$$\mathrm{v}^-(f) := \sup_{\kappa} \sum_{i=1}^{n} \left(f(t_i) - f(t_{i-1})\right)^- ,$$

we can further see that $\mathrm{v}(f) = \mathrm{v}^+(f) + \mathrm{v}^-(f)$ and $f(T) - f(0) = \mathrm{v}^+(f) - \mathrm{v}^-(f)$; the last two equalities imply $\mathrm{v}^+(f) = (\mathrm{v}(f) + f(T) - f(0))/2$. In combination with (2.14), this gives (2.12). $\square$

3 VOLATILITY OF CONTINUOUS PRICE PATHS

Let Ω' (our new *sample space*) be the set $C^+[0,T]$ of all positive continuous functions $\omega : [0,T] \to [0,\infty)$. Intuitively, this is the set of all possible price paths of our security. For each $t \in [0,T]$, the σ-algebra $\mathcal{F}'_t$ on Ω' is the trace of $\mathcal{F}_t$ on Ω' (i.e., $\mathcal{F}'_t$ consists of the sets $E \cap \Omega'$ with $E \in \mathcal{F}_t$). A *process* S is a family of functions $S_t : \Omega' \to [-\infty,\infty]$, $t \in [0,T]$, such that each S_t is $\mathcal{F}'_t$-measurable. A *simple capital process* is defined to be the restriction of a simple capital process in the old sense to Ω' (i.e., S' is called a simple capital process if there is a simple capital process S in the old sense such that, for each $t \in [0,T]$, $S'_t = S_t|_{\Omega'}$). *Positive capital processes* are capital processes S that can be represented in the form (2.2), where the simple capital processes $\mathcal{K}^{G_m}_t(\omega)$ are required to be positive, for all $t \in [0,T]$ and $\omega \in \Omega' = C^+[0,T]$, and the positive series $\sum_{m=1}^{\infty} c_m$ is required to converge, where c_m is the initial capital of G_m. (The notion of simple trading strategies does not change, but we are only interested in their behaviour on $\omega \in \Omega'$.) An *event* is an element of $\mathcal{F}'_T$. The definition of a null event is the same as before (but using the new notion of a positive capital process), and the adjective "typical" will again be used to refer to the complements of null events.

Remark 2. The definitions used in [21] are slightly different, but all proofs there also work under our current definitions. Under the assumption of continuity of ω, the requirement that ω should be positive is superfluous, and is never made in [21].

The following elaboration of Theorem 1 for continuous price paths was established in [20] using direct arguments (relying on the result in [1] mentioned earlier for the inequality $\mathrm{vi}(\omega) \le 2$ and a standard argument, going back to [7] and used in the context of mathematical finance in [16], Example 3 on p. 658, for the inequality $\mathrm{vi}(\omega) \ge 2$ for non-constant ω).

Proposition 3 [[20], Theorem 1]. *For typical $\omega \in \Omega'$,*

$$\mathrm{vi}(\omega) = 2 \textit{ or } \omega \textit{ is constant.}$$

This proposition is similar to the well-known property of continuous semimartingales (Lepingle [8], Theorem 1(a) and Proposition 3(b)). Related results in mathematical finance usually make strong stochastic assumptions (such as those in [13]). A probability-free result related to the inequality $\mathrm{vi}(\omega) \ge 2$ (for typical non-constant ω) was established by Salopek [14] (p. 228), who proved that the trader can start from 0 and end up with a strictly positive capital in a market with two securities whose price paths ω_1 and ω_2 are strictly positive, continuous, and satisfy $\mathrm{vi}(\omega_1) < 2$, $\mathrm{vi}(\omega_2) < 2$, $\omega_1(0) = \omega_2(0) = 1$, and $\omega_1(T) \ne \omega_2(T)$. However, Salopek's definition of a capital process only works under the assumption that all securities in the market have price paths ω satisfying $\mathrm{vi}(\omega) < 2$. The proof of Salopek's result was simplified in [11] (using the argument from [16] mentioned earlier).

The paper [21] establishes connections between continuous price paths and Brownian motion, which in combination with Taylor's [19] results greatly refine Proposition 3. Let $\psi : [0,\infty) \to [0,\infty)$ be Taylor's [19] function

$$\psi(u) := \frac{u^2}{2 \ln^* \ln^* u}, \tag{3.1}$$

with $\psi(0) := 0$ and $\ln^* u := 1 \vee |\ln u|$.

Proposition 4 [[21], Corollary 5]. *For typical $\omega \in \Omega'$,*

$$\mathrm{v}_\psi(\omega) < \infty.$$

Suppose $\phi : [0,\infty) \to [0,\infty)$ is such that $\psi(u) = o(\phi(u))$ as $u \to 0$. For typical $\omega \in \Omega'$,

$$\mathrm{v}_\phi(\omega) = \infty \textit{ or } \omega \textit{ is constant.}$$

A natural question related to Proposition 4 and Proposition 1 is whether $\mathrm{v}_\psi(\omega) < \infty$ holds for typical positive right-continuous functions ω.

Proposition 4 refines Proposition 3, but is further strengthened by the next result, Proposition 5. The following quantity was introduced by Taylor [19]: for $f : [0,T] \to \mathbb{R}$, set

$$\mathrm{w}(f) := \lim_{\delta \to 0} \sup_{\kappa \in K_\delta} \sum_{i=1}^{n} \psi\left(|f(t_i) - f(t_{i-1})|\right),$$

where K_δ is the set of all partitions $0 = t_0 \le \cdots \le t_n = T$ of $[0,T]$ whose mesh is less than δ: $\max_i(t_i - t_{i-1}) < \delta$. Notice that $\mathrm{w}(\omega) \le \mathrm{v}_\psi(\omega)$.

Proposition 5 [[21], Corollary 6]. *For typical $\omega \in \Omega'$,*

$$\mathrm{w}(\omega) \in (0,\infty) \text{ or } \omega \text{ is constant.}$$

4 THE CASE OF NO BORROWING

The definitions in this section are applicable both to the framework of Section 2 (where the sample space is the set Ω of all positive right-continuous functions on $[0,T]$, which we also denote $R^+[0,T]$ in this section) and to the framework of Section 3 (where the sample space is the set $\Omega' = C^+[0,T]$ of all positive continuous functions on $[0,T]$). In this paper, we only use positive capital processes S_t. However, even positive capital processes may involve borrowing cash or security: at each time, S_t is the price of a portfolio containing some amounts of security and cash; the total value of the portfolio is positive but nothing prevents either of its components to be strictly negative. In this section we consider a market where the trader is allowed to borrow neither cash nor security (borrowing security is essentially the same thing as short-selling in this context). Such markets have been considered by, e.g., Cover [2] and de Rooij and Koolen [3].

Let G be a simple trading strategy. As before, the components of G will be denoted c (the initial capital), τ_n (the stopping times), and h_n (the positions), and we imagine a trader who follows G. For $t \in [0,T]$ and $\omega \in \Omega$ or $\omega \in \Omega'$ (as appropriate), set $h_t(\omega) := h_n(\omega)$, where n is the unique number satisfying $t \in (\tau_n, \tau_{n+1}]$ (with $h_t(\omega) := 0$ if $t \le \tau_1(\omega)$); intuitively, h_t is the trader's position at time t. The amount of *cash* in the trader's portfolio at time t is defined to be $\mathcal{K}_t^G(\omega) - h_t(\omega)\omega(t)$. Let us say that the trading strategy G is *borrowing-free* if, for all ω and t, we have $h_t(\omega) \ge 0$ (the condition of *no borrowing security*) and $\mathcal{K}_t^G(\omega) - h_t(\omega)\omega(t) \ge 0$ (the condition of *no borrowing cash*). (Remember that being borrowing-free is a completely different requirement from being self-financing: all trading strategies considered in this paper are self-financing.)

It is easy to see that G is borrowing-free if and only if $c \ge 0$ and (2.13) is satisfied for all $n \in \{1,2,\dots\}$. Indeed, suppose the latter condition is satisfied. If $t \in [0,\tau_1(\omega)]$,

$$\mathcal{K}_t^G(\omega) - h_t(\omega)\omega(t) = c \ge 0.$$

And if $t \in (\tau_n(\omega), \tau_{n+1}(\omega)]$,

$$\mathcal{K}_t^G(\omega) - h_t(\omega)\omega(t) = \mathcal{K}_{\tau_n}^G(\omega) + h_n(\omega)(\omega(t) - \omega(\tau_n)) - h_n(\omega)\omega(t) = \mathcal{K}_{\tau_n}^G(\omega) - h_n(\omega)\omega(\tau_n) \ge 0.$$

In the framework of Section 2, where the sample space is $R^+[0,T]$, all trading strategies G for which $\mathcal{K}^G$ is positive are automatically borrowing-free (we already used this fact in the proof of Proposition 2). Indeed, let $\mathcal{K}^G$ be positive. If the condition of no borrowing security is violated and $h_t(\omega) < 0$, we can make $\mathcal{K}_t^G(\omega) < 0$ by modifying ω over $[t,T]$ and making $\omega(t)$ sufficiently large. (Intuitively, borrowing security is risky when its price can jump since there is no upper limit on the price.) If the condition of no borrowing cash is violated and $\mathcal{K}_t^G(\omega) - h_t(\omega)\omega(t) < 0$, we can make $\mathcal{K}_t^G(\omega) < 0$ by modifying ω over $[t,T]$ and setting $\omega(t) := 0$. (Intuitively, borrowing cash is risky when the security's price can jump since the price can drop to zero at any time.) We will see shortly that in the framework of Section 3, where the sample space is $C^+[0,T]$, the condition that G should be borrowing-free makes a big difference.

By a *borrowing-free capital process* we will mean a process S that can be represented in the form (2.2) where all trading strategies G_m are required to be borrowing-free and the positive series $\sum_{m=1}^{\infty} c_m$ is required to converge. This definition is applicable to the frameworks of both Section 2 and Section 3.

Let E be a set of positive continuous functions on $[0,T]$. Since $E \subseteq \Omega$ and $E \subseteq \Omega'$, *a priori* there are at least four natural definitions of the upper probability $\overline{\mathbb{P}}(E)$:

- $\overline{\mathbb{P}}_1(E)$ is the upper probability (2.11) with S ranging over the positive capital processes defined on the space $\Omega' = C^+[0,T]$ of all positive continuous functions on $[0,T]$;
- $\overline{\mathbb{P}}_2(E)$ is the upper probability (2.11), exactly as it is defined there; namely, S ranges over the positive capital processes defined on the space $\Omega = R^+[0,T]$ of all positive right-continuous functions on $[0,T]$;
- $\overline{\mathbb{P}}_3(E)$ is the upper probability (2.11) with S ranging over the borrowing-free capital processes defined on $\Omega' = C^+[0,T]$.
- $\overline{\mathbb{P}}_4(E)$ is the upper probability (2.11) with S ranging over the borrowing-free capital processes defined on $\Omega = R^+[0,T]$.

In fact, most of these definitions are equivalent:

Proposition 6. *For any set $E \subseteq C^+[0,T]$ of positive continuous functions on $[0,T]$,*

$$\overline{\mathbb{P}}_1(E) \le \overline{\mathbb{P}}_2(E) = \overline{\mathbb{P}}_3(E) = \overline{\mathbb{P}}_4(E).$$

There exists a set E of positive continuous functions on $[0,T]$ such that

$$\overline{\mathbb{P}}_1(E) = 0 < 1 = \overline{\mathbb{P}}_2(E) = \overline{\mathbb{P}}_3(E) = \overline{\mathbb{P}}_4(E).$$

Proof. The equality $\overline{\mathbb{P}}_2(E) = \overline{\mathbb{P}}_4(E)$ has already been demonstrated, and the equality $\overline{\mathbb{P}}_3(E) = \overline{\mathbb{P}}_4(E)$ is not difficult to prove. Therefore, $\overline{\mathbb{P}}_2(E) = \overline{\mathbb{P}}_3(E) = \overline{\mathbb{P}}_4(E)$. Now let E be the set of all $\omega \in C^+[0,T]$ such that $\mathrm{vi}(\omega) \in (0,2)$. According to Proposition 3, $\overline{\mathbb{P}}_1(E) = 0$. And according to Proposition 2, $\overline{\mathbb{P}}_2(E) = 1$: there are even individual elements $\omega \in E$ for which $\overline{\mathbb{P}}_2(\{\omega\})$ is arbitrarily close to 1 (such as $\omega(t) = 1 + \epsilon t$ for sufficiently small $|\epsilon|$). $\square$

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Appendix A The case of finite p-variation, $p > 2$

Let $p > 2$. Theorem 1 says that the trader can become infinitely rich when $\mathrm{v}_p(\omega) = \infty$. This appendix treats the case where $\mathrm{v}_p(\omega)$ is merely large, not infinitely large. We are now in the framework of Section 2: the sample space is $\Omega = R^+[0,T]$.

Proposition 7. *Let $p = 2 + \epsilon > 2$ and let $\delta > 0$. There is a positive capital process S such that $S_0 = 1$ and, for all $\omega \in \Omega$,*

$$S_T(\omega) > (1 - 2^{-\epsilon})(1 - 2^{-\delta})2^{-6-\epsilon-\delta}\frac{\mathrm{v}_{2+\epsilon}(\omega)}{(1 \vee \sup \omega)^{2+\epsilon+\delta}} - \frac{1}{4}. \tag{A.1}$$

Proof. In this proof we will see what the argument used in the proof of Theorem 1 gives in the case of a finite $\mathrm{v}_p(\omega)$. It will be convenient to modify the function $j(i)$ used in that argument, making it dependent on the given upper bound 2^L on ω. For $L \in \{0, 1, 2, \dots\}$, define $j_L(i)$ to be the smallest integer $j \geq 2 - L$ satisfying (2.7). This definition ensures that $2^{-j_L(i)} \geq \frac{1}{4}(\omega(t_i) - \omega(t_{i-1}))$ when $\sup \omega \leq 2^L$.

Fix temporarily $L \in \{0, 1, 2, \dots\}$. Now we set $w(j) := (1 - 2^{-\epsilon})2^{\epsilon(2-L)}2^{-\epsilon j}$, $j = 2 - L, 3 - L, \dots$; $(1 - 2^{-\epsilon})2^{\epsilon(2-L)}$ is the normalizing constant ensuring $\sum_{j=2-L}^{\infty} w(j) = 1$. Using the inequality between the two extreme terms in (2.8) (with the lower limit of summation $j = 2 - L$ instead of $j = 0$) and setting $S^{(L)} := 2^{1-L} S$, we obtain a positive capital process satisfying $S_0^{(L)} \leq 1$ and

$$
\begin{aligned}
S_T^{(L)}(\omega) = 2^{1-L} S_T(\omega) &\geq 2^{1-2L} \sum_{i \in I_+} w(j_L(i))(2^{-j_L(i)})^2 = 2^{1-2L}(1 - 2^{-\epsilon})2^{\epsilon(2-L)} \sum_{i \in I_+} (2^{-j_L(i)})^{2+\epsilon} \\
&\geq 2^{1-2L}(1 - 2^{-\epsilon})2^{\epsilon(2-L)}4^{-2-\epsilon} \sum_{i \in I_+} (\omega(t_i) - \omega(t_{i-1}))^{2+\epsilon}
\end{aligned}
$$

instead of (2.8)–(2.9). And instead of (2.10) we now obtain

$$
\begin{aligned}
\sup \omega \leq 2^L \Longrightarrow S_T^{(L)}(\omega) &\geq 2^{-2L}(1 - 2^{-\epsilon})2^{\epsilon(2-L)}4^{-2-\epsilon}\, \mathrm{v}_{2+\epsilon}(\omega) - 2^{-L} \sum_{j=2-L}^{\infty} w(j)2^{-j} \\
&> (1 - 2^{-\epsilon})2^{-2L-\epsilon L}4^{-2}\, \mathrm{v}_{2+\epsilon}(\omega) - \frac{1}{4}.
\end{aligned}
$$

Set $S := \sum_{L=0}^{\infty} (1 - 2^{-\delta})2^{-\delta L} S^{(L)}$ (recycling the notation S); $1 - 2^{-\delta}$ is the normalizing constant ensuring that the weights $(1 - 2^{-\delta})2^{-\delta L}$ sum to 1. For any ω and any upper bound $2^L \geq \sup \omega$, with $L \in \{0, 1, 2, \dots\}$, we will have

$$
S_T(\omega) \geq (1 - 2^{-\delta})2^{-\delta L} S_T^{(L)}(\omega) > (1 - 2^{-\epsilon})(1 - 2^{-\delta})4^{-2}(2^L)^{-2-\epsilon-\delta}\, \mathrm{v}_{2+\epsilon}(\omega) - \frac{1}{4}.
$$

Taking the L satisfying $1 \vee \sup \omega \leq 2^L < 2(1 \vee \sup \omega)$, we obtain

$$
S_T(\omega) > (1 - 2^{-\epsilon})(1 - 2^{-\delta})4^{-2}2^{-2-\epsilon-\delta}(1 \vee \sup \omega)^{-2-\epsilon-\delta}\, \mathrm{v}_{2+\epsilon}(\omega) - \frac{1}{4},
$$

which is equivalent to (A.1). □

Proposition 7 is mainly motivated by the case of discrete time. Suppose the trader is allowed to change his positions in ω only at times $0, T/N, 2T/N, \dots, T$ for a strictly positive integer N. This restriction is equivalent to replacing ω by $\omega_N \in R^+[0, T]$ defined by

$$
\omega_N(t) := \omega\left(\frac{T}{N}\left\lfloor \frac{N}{T} t \right\rfloor\right), \quad t \in [0, T].
$$

The discrete-time version of Proposition 7 (which is weaker than Proposition 7 itself) says that there is a positive capital process S such that $S_0 = 1$ and (A.1) holds for all elements of Ω of the form ω_N.