Lending, Lying, and Costly Auditing[*]

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Abstract

In this paper, we describe a bankruptcy game played in a pure-exchange, perfectly competitive economy, and establish the existence of competitive equilibria. The game admits of lying by borrowers and costly auditing by lenders. The equilibria are characterized by (endogenously determined) equilibrium probabilities of default, loan quantities, interest rates, and default risk premia, and by equilibria simultaneously determined in risk-free debt markets. We find that the optimal debt contract is the standard debt contract, and that the risk-free debt market may be inactive, as all parties may strictly prefer risky debt contracts to risk-free debt.

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1 Introduction

This paper investigates lending where borrowers have private information on their abilities to fulfill their obligations, and can strategically default. The setting we have in mind is one of private lending with possible bankruptcy. Our borrowers may be thought of as individuals, or perhaps as small private firms or entrepreneurs, whose financial success is difficult to monitor, and whose owners might therefore be able to attempt a fraudulent conveyance.

The study of lending with possible strategic default, or as we prefer, lying, and with costly auditing goes back at least to Townsend (1979). Influential models, such as that of Gale and Hellwig (1985), argue for the optimality of standard debt contracts, in which borrowers either fully repay their loans along with interest or, if incapable of doing so, turn over all they have to the lenders. In order to verify that everything indeed has been turned over, a lender could audit a borrower at cost. The model due to Gale and Hellwig, however, and the related model of credit rationing in Williamson (1986, 1987), require that a lender commit to auditing in all cases of default. Subsequent work considered the case of stochastic auditing; examples are Border and Sobel (1987), Townsend (1988), Bernanke and Gertler (1989), and Mookherjee and Png (1989). As Boyd and Smith (1994) show, stochastic auditing makes the standard debt contract suboptimal, at least among risk-neutral agents. However, they calibrate the loss from using standard debt contracts and show that it is small. A discussion of the impact of stochastic auditing on the optimal contract, and on the implications of precommitment to an audit strategy versus a requirement of time-consistency, is in Krasa and Villamil (2000).

The model of Border and Sobel, and the related papers cited above, have stochastic auditing written directly into the contract. This follows Townsend (1979), inasmuch as Townsend allows for preplay commitment to an audit policy. A different tradition focuses more on the costly audit itself, in which the frequency of auditing and of lying arise endogenously. Reinganum and Wilde (1986), for example, consider an exogenous tax code, which determines

payoffs, and solve mixed strategy Nash equilibrium audit and fraud probabilities. Similar models appear in Newman and Noel (1989), Khalil (1997), and Chatterjee et al. (2002); these are in the same general spirit as the time-consistency requirements of Krasa and Villamil.

The focus of the literature on costly auditing, then, has chiefly been on how frequently audits will occur, and what their impact on optimal debt instrument design is. The size of the debt, however, is exogenous. The usual assumption is that the borrower, imagined as an entrepreneur with an investment opportunity but without funding, has a fixed need of external financing. Any funding below this amount is useless, and the borrower never seeks funding beyond his needs.[1] So the borrower in this literature has a perfectly inelastic demand, and the lender, thought of as a bank or venture capitalist, is fully aware of the required amount, and generally does not try to negotiate any other loan size.

We drop this restriction, and have borrowers and lenders respond to interest rates that arise in competitive markets. This enables us to investigate the circumstances under which a costly state verification problem can arise; that is, we ask why a lender would offer a loan to a borrower who can strategically default, if it is costly to monitor the borrower. To answer this question, we solve for competitive equilibria in markets for risky and risk-free debt, where interest rates in all markets are endogenously determined, and where default probabilities, auditing probabilities, and penalties for lying are in equilibrium.

Our claim that we find equilibrium penalties for lying needs some qualification. We restrict attention to penalties that are proportionate to the debt, though we allow the penalties to depend on the borrower's credit worthiness. The limitation to proportionate penalties may seem restrictive, but it has a long pedigree: paragraph 112 of the Code of Hammurabi states,

> If any one be on a journey and entrust silver, gold, precious stones, or any movable
> property to another, and wish to recover it from him; if the latter do not bring
> all of the property to the appointed place, but appropriate it to his own use, then

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[1]The problem of a gambler in an analogous situation is famously considered in Dubins and Savage (1965).

shall this man, who did not bring the property to hand it over, be convicted, and
he shall pay fivefold for all that had been entrusted to him.

This penalty of quintupling the debt may seem severe by modern standards, but the standard debt contract, as mentioned above, is no less draconian. Borrowers pledge everything, so that a borrower who fraudulently defaults would presumably risk all he hopes to protect.

Our result differs from those of Boyd and Smith, in that they find that some debt forgiveness is optimal, although the latter show that the benefits of including debt forgiveness are small. Two of their remarks are noteworthy for us. First, Boyd and Smith state that, because of the limited ability to improve upon the standard debt contract in their setting, "even minor features of reality that are omitted from [their] model could easily render it optimal to employ standard debt contracts." Second, they subsequently point to risk-neutrality as a potentially crucial assumption: "our results (and particularly the characterization of an optimum that we borrow from Border and Sobel 1987) depend heavily on the assumption that agents are risk neutral." We show that their conjectures are correct: by changing to an environment with agents who have constant absolute risk-aversion, we recover the optimality of the standard debt contract. However, if the lender is risk-neutral, some debt forgiveness becomes optimal.

Endogenous lending and borrowing leads to some surprising results. The lender's preference need not be to lend to the borrower with the greatest ability to repay the debt. This is true even though a borrower's probability of lying is monotonically decreasing in his ability to repay. So while higher probabilities of good luck coincide with lower default rates, lenders may prefer a borrower with a higher default rate to one with a lower default rate.

A related result concerns auditing. In the model of Gale and Hellwig (1985) and related papers cited above, the goal is to minimize the deadweight loss due to costly auditing. In our setting, this is not the case in equilibrium. The lender will not in general prefer the contract with the lowest possible probability of strategic default, as noted above. This is because

our setting has multiple potential sources of inefficiency. Agents trade off the inefficiency of the deadweight loss from costly auditing against the potential efficiency gains from better smoothing across dates and across states. Nevertheless, we recover part of the result in Gale and Hellwig: the standard debt contract remains the optimal debt arrangement. But other mechanisms for reducing the need to audit, such as increasing loan size as a commitment device (as in Khalil and Parigi 1998) or decreasing loan size to ration credit, are not pushed to their extremes. If they were, the probability of strategic default would vanish, which can only happen in equilibrium in the risk-free debt market.

A surprising techincal result is that the lending supply, in both the risky and risk-free debt markets, is decreasing in the interest rate. That is, lending is a Giffen financial good. The reason is that, as interest rates increase, a risk-averse lender optimally smooths consumption by making a smaller loan. Our lenders have constant absolute risk-aversion, but similar results have appeared elsewhere with agents who have constant relative risk-aversion (e.g., Poulsen and Rasmussen 2008).

The structure of the rest of this paper is as follows: In the next section, we give the details of our game. In section 3, we derive equilibrium conditions on the probabilities of lying and auditing, given that an equilibrium with possible default exists. We then derive the equilibrium penalties for fraud. In section 4, we deriving the supply and demand curves of risky and risk-free lending, and then the default risk premia. We compare our results on the optimality of the standard debt contract to the findings in the literature in section 5, and give concluding remarks in section 6. All proofs are in Appendix A.

2   The Model

We consider a competitive pure-exchange economy with one non-storable good and two time periods. It is populated by a set of lenders $A = \{a_1, a_2, \ldots, a_N\}$ and a set of borrowers $B =$

$\{b_1, b_2, \ldots, b_N\}$.[2] The members of A are alike in all ways that matter, and the representative meber of A is a. We assume, although with no justification other than convenience, that a lends, if at all, to only one randomly selected member of B. That member is b, characterized by a certain ability to repay or credit worthiness, to be defined presently. We also assume that N is such that a and b, in solving their respective optimization problems, take the gross rate of interest R as given. Those assumptions allow us to restrict attention to the two agents a and b.

It is customary in the costly state verification literature for the loan size to be exogenous. Borrowers are thought of as entrepreneurs, needing some fixed level of external financing in order to undertake a project, while lenders are analogously viewed as venture capitalists, who might fund projects that they are unable to undertake themselves. Borrowers in these settings typically do not seek more funding than they need, and cannot find any use for amounts below the project's cost. By moving to a pure exchange setting, we are able to depart in the simplest possible way from the world of perfectly inelastic borrowing demand. And to keep things simple, we specify endowment vectors that identify lenders and borrowers before the beginning of play. The endowment vector of a is $(\omega_a, 0)$, where $\omega_a > 0$ is what she receives of the one good in our economy in the first period prior to trade; and 0 is her endowment in the second period. In sharp contrast, b's endowment vector is $(0, \widetilde{\omega}_b^j)$, $j \in \{L, H\}$, where 0 is what he receives in the first period; and $\widetilde{\omega}_b^j$ is a chance outcome. After the first period's endowments are received and any trade has taken place, Nature determines b's outcome:

$$\widetilde{\omega}_b = \begin{cases} \omega^H, & \text{with probability } \pi_b \\ \omega^L, & \text{with probability } 1 - \pi_b. \end{cases}$$

For us, that means that if $\pi_b > \pi_{b'}$, then b is the more credit worthy or the better credit risk than b'. The endowment vectors and probabilities are common knowledge, known to a and b before the start of play. Unless confusion can arise, we drop the subscript on π.

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[2]To avoid having to introduce a population of auditors (or a demand for and a supply of auditors), we assume that auditing requires only the application of a given number of units of the one good of our economy.

All agents in the economy are risk-averse, with preferences that are represented by a state-independent, time-separable utility function displaying constant absolute risk-aversion:

$$u(x) = 1 - e^{-\lambda x}, \text{ for } \lambda > 0. \tag{1}$$

If a chooses not to lend to b, her expected utility is

$$\overline{U}_a = 1 - e^{-\lambda \omega_a}, \tag{2}$$

and b's expected utility is

$$\overline{U}_b = 1 - \pi e^{-\lambda \omega^H} - (1-\pi)e^{-\lambda \omega^L}. \tag{3}$$

Ours is a game of incomplete information. Suppose that a lends $L > 0$ to b at gross interest rate $R > 0$. We assume that a does not observe the actual endowment that Nature gives to b. Instead, a receives a reported endowment $\widehat{\omega}_b$ from b. Another of our assumptions is that, when feasible, b can lie and, in contrast to models such as those of Ewert and Wagenhofer (2005) and Hau (2008), lying is costless. To clarify, if b's realized endowment in the second period is ω^L, where

$$\omega^L < R^* L^*, \tag{4}$$

then b cannot report $\widehat{\omega}_b^H$.[3] Were he to do so, he would obligate himself to do the impossible, repay a in full. Formally, $\Pr(\widehat{\omega}_b^H | \omega^L) = 0$. But having received the endowment ω^H, b can lie, and does with probability $p := \Pr(\widehat{\omega}_b^L | \omega^H)$.

As we also assume, though, a can audit b. As noted above, she does not commit to an audit policy before play has begun. Rather, having received b's report of his endowment, she audits him with probability $q := \Pr(\text{audit} | \widehat{\omega}_b^L)$. The audit cost is $\delta \in (0, \omega^L)$. By assumption, a's audit technology is perfect. She has only to pay the cost δ to learn b's realized endowment.

Any loan contract a offers to b contains (1) a loan amount $L \in (0, \omega_a]$ for each admissible R. It also contains (2) certain payment provisions: (i) if b can pay what he promised or is

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[3]For any endogenously determined variable y, y^* is its equilibrium value.

discovered to be able to repay a in full, he must (and does); (ii) if b cannot repay in full, a takes what wealth b has; and (iii) if b is caught out in an attempt to deceive a, he must not only repay a in full, but in addition pay a a penalty γRL, where $\gamma > 0$ is endogenously determined. To accommodate the possibility of b lying and being caught out, we add another restriction,

$$\omega^H \geq (1 + \gamma^*)R^*L^*, \tag{5}$$

which must hold in equilibrium. The payments provisions (i)-(iii), taken together, constitute an exogenously imposed bankruptcy code.[4]

In the first period, upon receiving her endowment, a decides whether to offer a loan contract to b. She has several options. She may decide to do no lending, and if so there is no further play. She may decide to offer a risk-free loan contract to b, so that, given risk-free gross interest rate R_f and loan size L_f, the repayment satisfies $R_f L_f \leq \omega^L$. In other words, a risk-free loan will not satisfy inequality (4). Or she may offer a risky loan contract to b, where (4) holds.

If a offers a loan contract to b, the latter may accept or reject it. More exactly, b takes the interest rate as given, and calculates his borrowing demand at the going interest rate. So acceptance of the contract requires that the amount of funds that a is willing to lend equals b's borrowing demand, i.e., that the interest rate is a competitive equilibrium rate and that the loan is the equilibrium debt size.

In the second period, Nature then chooses b's endowment and, if the loan contract is risky, a and b then play the audit game described above: b issues report $\widehat{\omega}_b$, and a chooses whether to audit or trust b's report. Should the contract instead be for a risk-free loan, we assume that a never audits b, as b must repay the loan with interest in full in both states of the world. We illustrate the timeline of the game here.

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[4]For codes that are determined endogenously, see Grochulski (2005).

a receives ω_a a lends L^* to b   b receives $\widetilde{\omega}_b$ reports $\widehat{\omega}_b$ audit?   repayment

0   1

3   Endogenous Probabilities and Penalties

3.1   Equilibrium probabilities of lying and auditing

As the timeline and discussion in Section 1 indicate, the lender and borrower are only in a competitive market before the loan is agreed upon. Once a loan contract has been offered and accepted, the lender a and borrower b enter into a two-player game, where b chooses between reporting his endowment truthfully and lying, and a chooses whether to audit or trust b's report.

There are no pure strategy Nash equilibria for our game. For suppose that b always tells the truth. Then, knowing that and not having made an initial commitment to an audit strategy, a never audits. So, whenever feasible, b optimally lies. Or suppose that, if feasible, b lies. Then a optimally always audits, and in consequence b always tells the truth. In equilibrium, then, b must mix. An analogous argument shows that a must also mix. We now derive expressions for the equilibrium lying probability p^* and auditing probability q^*.

Suppose there is risky lending in equilibrium, and that the loan size is L^* at gross interest rate R^*. Then, given (p, q), a's expected utility is

$$
\begin{aligned}
EU_a &= u(\omega_a - L^*) + \pi(1-p)u(R^*L^*) + (1 - \pi + \pi p)(1-q)u(\omega^L) \qquad (6)\\
&+ \pi pqu((1+\gamma^*)R^*L^* - \delta) + (1-\pi)qu(\omega^L - \delta).
\end{aligned}
$$

We can decompose equation (6) as follows. The first term represents a's first period consumption. The second term is a's expected utility from b's receiving the high endowment

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and choosing to honor the debt. The third is a's expected utility from not auditing when b defaults. The fourth and fifth are a's expected utility from auditing when b defaults, either from catching b lying or from verifying that b reported truthfully.

Similarly, b's expected utility from accepting a risky loan L^* at interest rate R^* is

$$
\begin{aligned}
EU_b \;=\;& u(L^*) + \pi[(1-p)u(\omega^H - R^*L^*) \\
& + p(1-q)u(\omega^H - \omega^L) + pqu(\omega^H - (1+\gamma^*)R^*L^*)].
\end{aligned}
\tag{7}
$$

We can find (p^*, q^*) directly from (6) and (7); for similar results, see e.g. Reinganum and Wilde (1986) or Newman and Noel (1989):

Proposition 1 *Given* $(\pi, \lambda, \omega_a, \omega^H, \omega^L, \delta)$ *and the equilibrium values* (γ^*, L^*, R^*), *if inequality (4) holds, then there are unique Nash equilibrium values of* p^* *and* q^*. *In particular,*

$$
q^* = \frac{e^{\lambda R^* L^*} - e^{\lambda \omega^L}}{e^{\lambda(1+\gamma^*)R^* L^*} - e^{\lambda \omega^L}}
\tag{8}
$$

and

$$
p^* = \left(\frac{1-\pi}{\pi}\right) \cdot \left(\frac{(e^{\lambda\delta} - 1)e^{\lambda(1+\gamma^*)R^* L^*}}{e^{\lambda(1+\gamma^*)R^* L^*} - e^{\lambda(\omega^L+\delta)}}\right).
\tag{9}
$$

The form of equation (9) raises the possibility that, for sufficiently low values of π (i.e., for borrowers with sufficiently poor credit worthiness), p^* may be greater than or equal to 1. Since there is no equilibrium in pure strategies, this would seem to suggest a cutoff level $\underline{\pi}$ below which only risk-free borrowing or autarky could be equilibrium. However, p^* is derived in Proposition 1 given (γ^*, L^*, R^*). Hence to discuss whether p^* is necessarily in $(0,1)$, or if for certain values of π the risky debt market shuts down, we now turn to calculating the equilibrium default penalties.

3.2 Equilibrium penalty parameters

Our calculation of γ^* uses an envelope theorem argument. In any equilibrium where inequality (4) holds, equation (7) reduces to

$$EU_b = u(L^*) + \pi u(\omega^H - R^*L^*), \tag{10}$$

which is independent of γ^*. The intuition is as follows: since markets are competitive, b assumes his borrowing demand L^D has no influence on R^*. As long as a audits optimally given γ, b is indifferent between lying and reporting honestly when $\widetilde{\omega}_b = \omega_b^H$. So in an equilibrium, b's utility is independent of γ.

For a, the situation is different. In an equilibrium where inequality (4) holds, equation (6) becomes

$$EU_a = u(\omega_a - L^*) + \pi(1 - p^*)u(R^*L^*) + (1 - \pi + \pi p^*)u(\omega^L). \tag{11}$$

Since a must be indifferent between auditing and trusting a report of $\widehat{\omega}_b^L$, her expected utility in equilibrium will be equal to her utility of trusting b's report. Accordingly, the possibility of b lying affects a's utility, and since p^* depends on γ, EU_a will also depend on γ.

Proposition 2 (Optimality of Standard Debt Contract) *In an equilibrium with risky debt, given $(\pi, \lambda, \omega_a, \omega^H, \omega^L, \delta)$ and given (L^*, R^*), the unique optimal penalty parameter is*

$$\gamma^* = \frac{\omega^H - R^*L^*}{R^*L^*} = \frac{\omega^H}{R^*L^*} - 1.$$

In other words, the penalty is set so that, if there is default, the lender takes everything.

Observe that Proposition 2 means that the contract is essentially the standard debt contract. The borrower agrees to pay the principal plus interest, and if he defaults, then the lender claims everything the borrower has. If an audit establishes that b has retained any resources after having defaulted, then all of those resources are to be turned over to a. As remarked upon in the introduction, this result differs from those in Boyd and Smith (1994), who work

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in a setting of risk-neutral lenders and risk-neutral borrowers. We discuss this contrast below in Section 5.

Propositions 1 and 2 have the following, somewhat Brechtian, corollary:

Corollary 3 *In an equilibrium with risky debt, less credit worthy borrowers are more dishonest; i.e., p^* decreases in π. The audit probability q^* does not depend on the borrower's credit worthiness or honesty, except insofar as these affect the amount to be repaid.*

We also have the following:

Corollary 4 *There is a unique level of credit worthiness $\underline{\pi} < 1$ such that there can be no equilibrium in risky debt for $\pi \leq \underline{\pi}$.*

We end this subsection with some properties of the equilibrium probabilities.

Corollary 5 *The equilibrium auditing probability q^* is strictly decreasing in λ, ω^H, and ω^L. It is strictly increasing in R^*L^*.*

The equilibrium lying probability p^* is strictly increasing in λ, δ, and ω^L, and strictly decreasing in ω^H.

4   Equilibrium Loan Size and Interest Rates

4.1   Loan size

The analysis so far solves for the Nash equilibrium in the subgame between the lender and borrower, once the lending arrangement is agreed upon, and for the equilibrium penalty

for fraud. We now turn to finding the equilibrium values of the loan size. We begin by considering borrower b's demand, given that he is willing to take on risky debt.

From equation (10) and the envelope theorem, we obtain the first-order condition

$$\frac{\partial EU_b}{\partial L} = \lambda e^{-\lambda L^D} - \lambda R\pi e^{-\lambda \omega^H} e^{\lambda R L^D} = 0,$$

where $L^D = L^D(R)$ is the optimal risky loan size, given that b chooses to take on risky debt and given gross interest rate R. Rearranging gives the following loan demand function:

$$L^D(R) = \frac{\lambda \omega^H - \ln(\pi R)}{\lambda(1 + R)}. \tag{12}$$

Proposition 6 *Given that b's participation constraint is satisfied and that b prefers risky debt to risk-free debt, L^D strictly increases in ω^H and strictly decreases in π and R. The change in L^D with respect to λ depends on the size of R: demand increases (respectively decreases) in λ if and only if $R > 1/\pi$ (respectively $R < 1/\pi$).*

The borrowing demand as described in (12) is incomplete, because at any interest rate R, b may choose to borrow a risk-free level L_f such that (4) does not hold, i.e., such that $RL_f \leq \omega^L$. In this case, b's expected utility becomes

$$\widetilde{EU}_b = u(L_f) + \pi u(\omega^H - RL_f) + (1 - \pi)u(\omega^L - RL_f). \tag{13}$$

The first-order condition with respect to L_f is then

$$\frac{\partial \widetilde{EU}_b}{\partial L_f} = u'(L_f^D) - R\left[\pi u'(\omega^H - RL_f^D) + (1 - \pi)u'(\omega^L - RL_f^D)\right] = 0,$$

which upon substitution yields

$$L_f^D(R) = -\frac{\ln\left(R\left[\pi e^{-\lambda \omega^H} + (1 - \pi)e^{-\lambda \omega^L}\right]\right)}{\lambda(1 + R)}. \tag{14}$$

Observe that b cannot simultaneously enter the risky and risk-free debt markets: once the face value of b's debt exceeds ω^L, any debt b takes on is risky. That is, we are abstracting

from borrowing from multiple sources, in order to avoid having to consider questions of absolute priority and related matters.

Figure 1 illustrates the demand for risk-free and risky borrowing at given interest rates. At any given interest rate, the borrower would prefer a larger loan if it is risky than if it is risk-free; this is because the risky loan at a given rate is not always repaid in full.
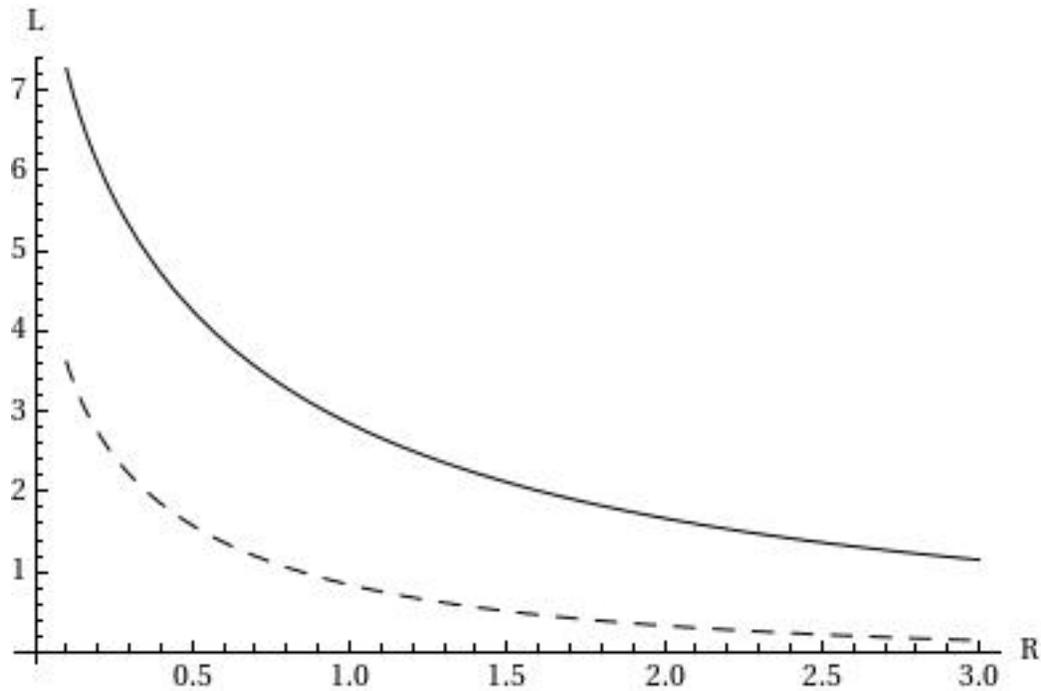

Figure 1: Inverse borrowing demand for risky debt (solid) and risk-free debt (dashed).

The borrower will choose a loan size at a given interest rate that maximizes his expected utility. The borrowing demand function is therefore

$$\text{Borrowing demand} = \begin{cases} L^D, & \text{if } EU_b(L^D(R)) = \max\left\{\overline{U}_b, EU_b(L^D(R)), \widetilde{EU}_b(L_f^D(R))\right\} \\ L_f^D, & \text{if } \widetilde{EU}_b(L_f^D(R)) = \max\left\{\overline{U}_b, EU_b(L^D(R)), \widetilde{EU}_b(L_f^D(R))\right\} \\ 0, & \text{otherwise.} \end{cases}$$

(15)

Borrowing demand is piecewise normal, but it may observationally appear that the borrowing demand is Giffen. As R nears a critical value, the borrowing demand can jump as the borrower moves from preferring the risk-free loan to the risky loan. This discontinuity can make the borrowing demand increase in the interest rate, but only near values where b's utility from the risky loan is close to his utility from the risk-free loan.

Turning now to the lender, we begin by assuming that a enters the risky debt market, and calculate her lending supply. Then we calculate her supply of risk-free lending, after which we can determine the lending equilibrium.

The payoff to a from risky debt depends on the likelihood of default. From (11) and the envelope theorem, we obtain the first-order condition

$$\frac{\partial EU_a}{\partial L} = -u'(\omega_a - L^S) + R\pi(1 - p^*)u(RL^S) = 0$$
$$\Rightarrow -\lambda e^{-\lambda \omega_a} e^{\lambda L^S} + \lambda R\pi(1 - p^*)e^{-\lambda R L^S} = 0,$$

where we have used the fact that p^* is independent of L (from equation (32) in Appendix A). Rearranging and substituting gives a's supply of risky debt, given that she enters the risky debt market:

$$L^S(R) = L^S = \frac{\lambda \omega_a + \ln\left\{R\pi\left[1 - \left(\frac{1-\pi}{\pi}\right)\left(\frac{e^{\lambda(\omega^H + \delta)} - e^{\lambda \omega^H}}{e^{\lambda \omega^H} - e^{\lambda(\omega^L - \delta)}}\right)\right]\right\}}{\lambda(1 + R)}$$
$$= \frac{\lambda \omega_a + \ln\left[R\pi(1 - p^*)\right]}{\lambda(1 + R)}. \quad (16)$$

The following properties are noteworthy:

Proposition 7 *Given that a's participation constraint is satisfied and that she prefers risky debt to risk-free debt, L^S strictly increases in π, ω_a and ω^H, and strictly decreases in ω^L. It is non-monotonic in R.*

Figure 2 shows the behavior of the lending supply for risky debt as a function of the interest rate. As R increases, L^S goes from being normal to being downward sloping in R. Contrast this with L^D, which shows piecewise normal behavior.
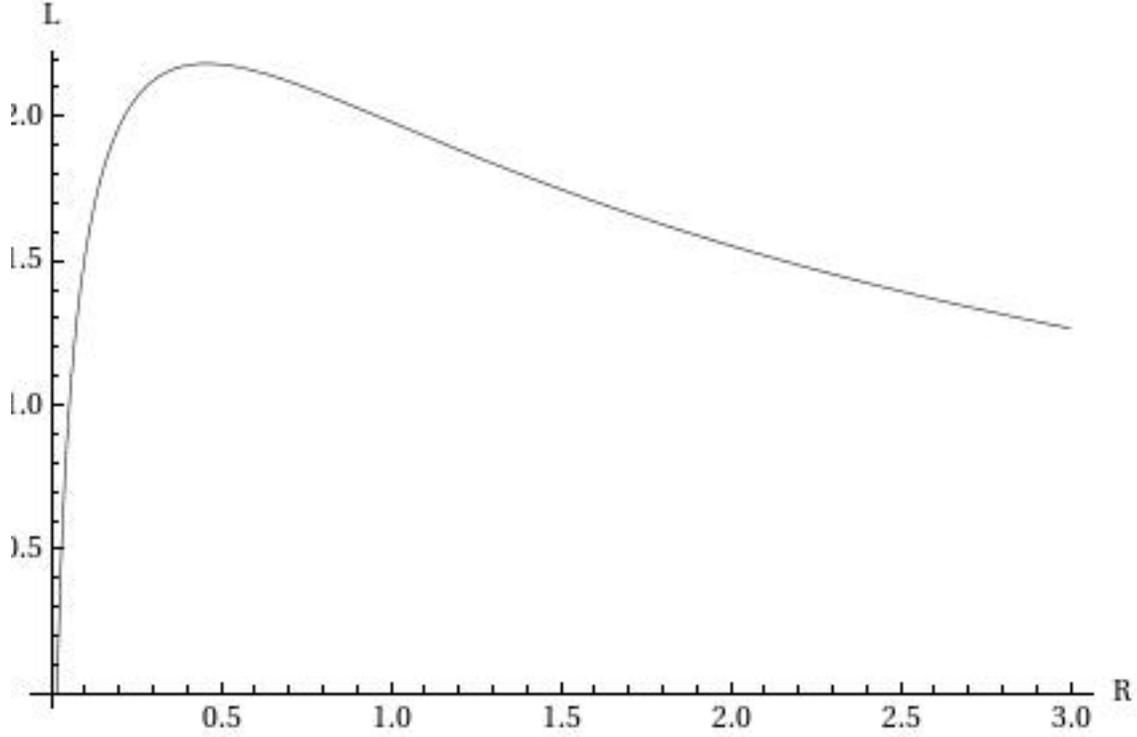
Figure 2: Inverse supply of risky debt. Although it is normal at first, the inverse supply function becomes downward sloping for sufficiently high R.

Proposition 7 is not a result of the costly auditing issue. Instead, it appears to be a consequence of having a lender with CARA utility. We show this initially by considering the first-best case, in which b cannot feasibly lie, and then by turning to the risk-free debt market, in which the informational asymmetry is of no importance. In each case, the inverse lending supply function is eventually downward-sloping.

The first-best case is when the borrower's endowment is publicly observable, i.e., when $\delta = 0$. In that case, it would be pointless (and costly) for b to lie, and consequently a's expected utility becomes

$$EU_a^{FB} = u(\omega_a - L_{FB}) + \pi u(RL_{FB}) + (1 - \pi)u(\omega^L). \tag{17}$$

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Differentiating with respect to the loan size gives

$$\frac{\partial EU_a^{FB}}{\partial L_{FB}} = -u'(\omega_a - L_{FB}^S) + R\pi u'(RL_{FB}),$$

which upon rearranging and substitution gives the first-best lending supply

$$L_{FB}^S(R) = \frac{\lambda\omega_a + \ln(R\pi)}{\lambda(1+R)}. \tag{18}$$

Comparing (16) with (18), one easily sees that, at any given gross interest rate R,

$$L_{FB}^S(R) = L^S(R) - \frac{\ln(1-p^*)}{\lambda(1+R)}.$$

Since $\ln(1-p^*) < 0$ and p^* is independent of R by Proposition 2, L_{FB}^S asymptotically approaches L^S from above as R increases. Since L^S is eventually decreasing in R, the first-best lending supply L_{FB}^S must also decrease.

We now consider the risk-free debt market. When facing gross interest rate R, a may choose only to supply a loan to b that she knows b can repay in every state of the world. Upon making such a risk-free loan, a obtains expected utility of

$$\widetilde{EU}_a = u(\omega_a - L_f) + u(RL_f). \tag{19}$$

Her first-order condition with respect to L_f is then

$$\frac{\partial \widetilde{EU}_a}{\partial L_f} = -u'(\omega_a - L_f^S) - Ru'(RL_f^S) = 0,$$

which upon substitution yields

$$L_f^S(R) = \frac{\lambda\omega_a + \ln R}{\lambda(1+R)}. \tag{20}$$

Comparing (18) with (20), one easily sees that, at any given gross interest rate R,

$$L_f^S(R) = L_{FB}^S(R) - \frac{\ln\pi}{\lambda(1+R)} = L^S(R) - \frac{\ln(1-p^*)}{\lambda(1+R)} - \frac{\ln\pi}{\lambda(1+R)}. \tag{21}$$

Proposition 8 *The lending supply function is eventually decreasing in R.*

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The downward-sloping lending supply curve may be surprising, but a uses the credit market to smooth across dates and across states. An increase in R increases a's overall wealth. For large values of R, a's expected second-period consumption approaches its asymptotic bound, as utility in each period is bounded above at 1. Eventually, the increase in income is better substituted into current consumption, as lending generates little incremental utility.

Although the boundedness of CARA utilities suffices to make lending Giffen, it is not necessary. Poulsen and Rasmussen (2008) show the existence of financial Giffen goods in a variation of a portfolio selection problem due to Merton (1990), in which agents have constant relative risk-averse utility with a minimum wealth requirement. For Poulsen and Rasmussen, a financial Giffen good is an asset whose weight in a portfolio decreases as its expected return goes up; they show that the the risk-free asset is Giffen. Even without a subsistence requirement, lending can be Giffen for agents with constant relative risk-aversion (though not in the Merton portfolio choice problem). In Diamond's (1965) national debt model, agents have CRRA utility, yet savings may decrease in wealth (see the discussion in Romer 2005, 79–80).

Figure 3 illustrates the inverse lending supply for risk-free and risky borrowing at given gross interest rates.

The lender will offer a loan size at a given interest rate that maximizes her expected utility. Her supply function is therefore

$$
\text{lending supply} = \begin{cases} L^S, & \text{if } EU_a(L^S(R)) = \max\{\overline{U}_a, EU_a(L^S(R)), \widetilde{EU}_a(L^S(R))\} \\ L_f^S, & \text{if } \widetilde{EU}_a(L_f^S(R)) = \max\{\overline{U}_a, EU_a(L^S(R)), \widetilde{EU}_a(L_f^S(R))\} \\ 0, & \text{otherwise.} \end{cases} \tag{22}
$$

4.2   Equilibrium loan rates

Determination of equilibrium is now straightforward. For the risky debt market to be in competitive equilibrium and for there to be no incentives for entry or exit, we must have
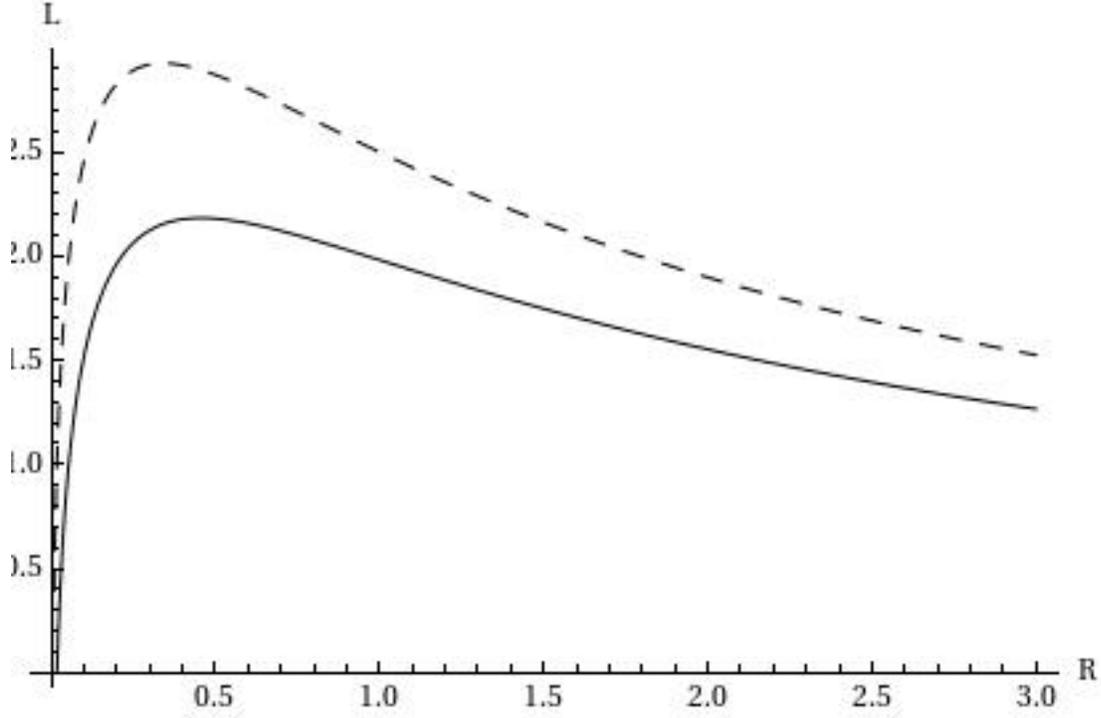
Figure 3: Inverse lending supply for risky debt (solid) and risk-free debt (dashed) as a function of the interest rate.

$L^S(R) = L^D(R)$. This gives us our main existence and uniqueness result:

Theorem 9 *There is a unique equilibrium in the market for risky debt, with*

$$R^* = \left[\frac{e^{\lambda(\omega^H - \omega_a)}}{\pi^2 (1 - p^*)} \right]^{\frac{1}{2}}, \tag{23}$$

where

$$p^* = \left(\frac{1 - \pi}{\pi} \right) \cdot \left(\frac{e^{\lambda(\omega^H + \delta)} - e^{\lambda \omega^H}}{e^{\lambda \omega^H} - e^{\lambda(\omega^L + \delta)}} \right),$$

provided π is sufficiently large. The market may be inactive, but the equilibrium size of the risky debt (whether latent or observed) is

$$L^* = \frac{\pi \sqrt{1 - p^*} \left[\lambda \left(\omega^H + \omega_a \right) + \ln \left(1 - p^* \right) \right]}{2\lambda \left[\pi \sqrt{1 - p^*} + e^{\lambda(\omega^H - \omega_a)/2} \right]},$$

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i.e.,

$$L^* = \frac{\lambda(\omega^H + \omega_a) + \ln(1 - p^*)}{2\lambda(1 + R^*)}.$$ (24)

The remaining parameters are given by

$$q^* = \frac{e^{\lambda R^* L^*} - e^{\lambda \omega^L}}{e^{\lambda \omega^H} - e^{\lambda \omega^L}} \qquad \gamma^* = \frac{\omega^H - R^* L^*}{R^* L^*}.$$

In light of Corollaries 3 and 5, we have the following:

Corollary 10 *The equilibrium gross interest rate R^* in the risky debt market is increasing in ω^L and δ. It is decreasing in π and in ω_a. The effects of ω^H and λ are ambiguous.*

Whether the market is active is determined by both parties' participation constraints and by the equilibrium in the risk-free debt market. Again, equilibrium is determined by setting $L_f^S(R) = L_f^D(R)$. We obtain an analogous result to Theorem 9.

Theorem 11 *There is a unique equilibrium in the risk-free debt market, with*

$$R_f = \left[\frac{e^{-\lambda \omega_a}}{\pi e^{-\lambda \omega^H} + (1 - \pi)e^{-\lambda \omega^L}} \right]^{\frac{1}{2}} = \left[\frac{1 - \overline{U}_a}{1 - \overline{U}_b} \right]^{\frac{1}{2}}.$$ (25)

The market may be inactive, but the equilibrium size of the risk-free debt (whether latent or observed) is

$$
\begin{aligned}
L_f &= \frac{\left(\lambda \omega_a - \ln\left[\pi e^{-\lambda \omega^H} + (1 - \pi)e^{-\lambda \omega^L} \right] \right) e^{\lambda \omega_a} \left[\pi e^{-\lambda \omega^H} + (1 - \pi)e^{-\lambda \omega^L} \right]}{2\lambda e^{\lambda \omega_a} \left[\pi e^{-\lambda \omega^H} + (1 - \pi)e^{-\lambda \omega^L} + 1 \right]} \\
&= \frac{\lambda \omega_a - \left[\pi e^{-\lambda \omega^H} + (1 - \pi)e^{-\lambda \omega^L} \right]}{2\lambda(R_f + 1)}
\end{aligned}
$$ (26)

From (25), we see that the risk-free gross interest rate is determined by the distance each party is from the upper bound on utility in the period in which endowments are received.

We have following reassuring result.

Proposition 12 *The equilibrium risk-free interest rate is always smaller than the equilibrium risky interest rate. The two converge as π approaches 1.*

We end this subsection with some illustrations.

Example 13 *Let $(\pi, \lambda, \omega_a, \omega^H, \omega^L, \delta) = (0.9, 0.3, 10, 15, 5, 0.03)$. Then the equilibrium values are:*

$$R^* \approx 2.353 \quad L^* \approx 3.727 \quad p^* \approx 0.001 \quad q^* \approx 0.110 \quad \gamma^* \approx 0.710 \quad R_f \approx 1.241 \quad L_f \approx 2.207.$$

The lender's reservation utility is $\overline{U}_a \approx 0.950$, while her expected utility from the risk-free loan is $\widetilde{EU}_a \approx 1.464$ and her expected utility from the risky loan is $EU_a \approx 1.760$. So a wishes to trade, and prefers to enter the risky debt market.

The borrower's reservation utility is $\overline{U}_b \approx 0.992$, while his expected utility from the risk-free loan is $\widetilde{EU}_b \approx 1.411$ and his expected utility from the risky loan is $EU_b \approx 1.434$. So b wishes to trade, and also prefers to enter the risky debt market.

Thus, even though both parties are risk-averse, both prefer risky debt to risk-free debt, and both prefer trade to autarky.

The next example shows an equilibrium where the risk-free debt market is active and the risky debt market is latent. The only change in the parameters is that π is now 0.5 rather than 0.9:

Example 14 *Let $(\pi, \lambda, \omega_a, \omega^H, \omega^L, \delta) = (0.5, 0.3, 10, 15, 5, 0.03)$. Then the equilibrium values are:*

$$R^* \approx 4.254 \quad L^* \approx 2.376 \quad p^* \approx 0.010 \quad q^* \approx 0.190 \quad \gamma^* \approx 0.484 \quad R_f \approx 0.652 \quad L_f \approx 2.908.$$

The lender's reservation utility is $\overline{U}_a \approx 0.950$, while her expected utility from the risk-free loan is $\widetilde{EU}_a \approx 1.315$ and her expected utility from the risky loan is $EU_a \approx 1.762$. So a wishes to trade, and prefers to enter the risky debt market.

The borrower's reservation utility is $\overline{U}_b \approx 0.942$, while his expected utility from the risk-free loan is $\widetilde{EU}_b \approx 1.375$ and his expected utility from the risky loan is $EU_b \approx 0.894$. So b is unwilling to enter the risky debt market, but he will trade in the risk-free debt market.

Examples 13 and 14 show that the credit worthiness of a borrower b, π_b, is not always a good indicatior of the borrower's likely default rate, even though Corollary 3 shows that poorer credit risks also strategically default more often. The default risk premia that arise in equilibrium are sufficiently high to deter borrowers with low credit scores from entering the market, even when there are lenders who would prefer offering them risky loans to loans they can afford to repay.

Both markets can simultaenously shut down. For example, if π is sufficiently low, then by Corollary 4, there can be no equilibrium in the risky debt market. And if, in addition, ω_a is sufficiently small, then it is easy to see from (26) that the amount of risk-free debt offered in equilibrium vanishes or becomes negative.

We now show that, even when both parties prefer to be in the risky debt market, the lender may prefer a borrower with a lower credit worthiness π to one with a higher credit worthiness $\pi' > \pi$:

Example 15 *Consider Example 13 again, but now suppose that $\pi = 0.95$ instead of 0.9. Both parties continue to prefer risky debt, with the equilibrium gross interest rate dropping to $R^*_{0.95} \approx 2.229$ from $R^*_{0.9} \approx 2.353$ and the equilibrium loan size increasing to $L^*_{0.95} \approx 3.871$ from $L^*_{0.9} \approx 3.727$.*

The expected utility to a from lending to the higher credit worth borrower, with $\pi = 0.95$, is $EU_a \approx 1.758$, while that of lending to the lower credit worth borrower with $\pi = 0.9$ is $EU_a \approx 1.760$. Thus, the lender would prefer the less credit worthy borrower. It is also the case that both parties, in both examples, prefer the risky loan.

Our final example shows that the a does not necessarily prefer a less expensive audit to a more expensive audit, even if both audits are perfect:

Example 16 *Suppose, as in Example 14, that*

$$(\pi, \lambda, \omega_a, \omega^H, \omega^L) = (0.5, 0.3, 10, 15, 5),$$

but that auditing is now much more costly. Assume two audit technologies are feasible: $\delta_1 = 2.10$ *and* $\delta_2 = 2.12$.

Under the first audit technology,

$$R_1^* \approx 23.710 \quad L_1^* \approx 0.273 \quad R_1^* L^* 1 \approx 6.484 \quad p^* \approx 0.968 \quad q^* \approx 0.029,$$

while under the second audit technology,

$$R_2^* \approx 30.866 \quad L_2^* \approx 0.184 \quad R_2^* L^* 2 \approx 10.111 \quad p^* \approx 0.981 \quad q^* \approx 0.012.$$

The lender's expected utility from the autarky remains $\overline{U}_a \approx 0.950$, *and her expected utility from the risk-free loan remains* $\widetilde{EU}_a \approx 1.315$. *Her expected utility from the risky loan under the cheaper audit technology is* $EU_{a,1} \approx 1.724$, *and her expected utility from the risky loan under the more expensive audit technology is* $EU_{a,2} \approx 1.725$. *Hence, her first choice is to make the risky loan with the more expensive auditing technology, even though she will audit less often and even though b will lie more often.*

The intuition behind Example 16 is straightforward: as auditing becomes more costly, a becomes less likely to engage in the audit, and accordingly b becomes more willing to lie. This drives the equilibrium interest rate up, and the gain in interest on the occasions when b tells the truth can be enough to offset the higher auditing cost.

4.3   Default risk premia

There are two sources of premia in interest rates: the intrinsic risk π of b receiving a low endowment, and strategic risk that, given a good endowment, b will lie and attempt to keep

more for himself. We decompose the default risk premia into these components.

From (12) and (18), we obtain first-best value for R

$$L^D(R_{FB}) = L^S_{FB}(R_{FB}) = L_{FB} \Rightarrow \frac{\lambda\omega^H - \ln(R_{FB}\pi)}{\lambda(R_{FB}+1)} = \frac{\lambda\omega_a + \ln(R_{FB}\pi)}{\lambda(R_{FB}+1)}$$

$$\Rightarrow \ln(R_{FB}) = \frac{1}{2}\left[\lambda(\omega^H - \omega_a) - 2\ln\pi\right]$$

$$\Rightarrow R_{FB} = \left[\frac{e^{\lambda(\omega^H - \omega_a)}}{\pi^2}\right]^{\frac{1}{2}} \tag{27}$$

Proposition 17 *The default risk premium due to strategic risk is given by the following:*

$$\frac{R^*}{R_{FB}} = \frac{1}{\sqrt{1-p^*}}. \tag{28}$$

The default risk premium due to intrinsic risk is

$$\frac{R_{FB}}{R_f} = \frac{\sqrt{\pi + (1-\pi)e^{\lambda(\omega^H - \omega^L)}}}{\pi}. \tag{29}$$

The total default risk premium is therefore

$$\frac{R^*}{R_f} = \left[\frac{\pi + (1-\pi)e^{\lambda(\omega^H - \omega^L)}}{\pi^2(1-p^*)}\right]^{\frac{1}{2}}.$$

Plugging R_{FB} into the first-best lending supply function (or into the lending demand function) for risky debt then gives

$$L_{FB} = \frac{\omega^H + \omega_a}{2\lambda(R_{FB}+1)} = \lim_{p^* \nearrow 1} L^*. \tag{30}$$

We now use R_{FB} and L_{FB} to gain further insight into Example 15, where the lender prefers higher default probability to lower default probability. Under the parameters in Example 13, the expected utility to a of the first-best loan is $EU_a^{FB} \approx 1.586$, while her expected utility of the risky loan is $EU_a \approx 1.760$. This still exceeds her expected utility from the risk-free debt, but the possibility of lying drives the loan size down and the interest rate up. Because p^* is

very low (below 1%) in this example, the gain from the higher interest rate and lower initial outlay is enough to offset the small increment in strategic risk from lending to a strategic borrower over lending to a compulsively honest borrower. The effect is that a has to give up less in the first period and, with very high likelihood, gains more in the second.

The fact that a can prefer strategic default to honesty conflicts with the insights in Townsend (1979) and Gale and Hellwig (1985). In those papers, the lender always precommits to auditing with certainty, in order to prevent the borrower from strategically defaulting. Others have questioned whether the lender can always commit to an audit strategy (e.g. Khalil 1997). Example 15 and the discussion above raise an additional possibility: even when the lender can commit to an audit policy, it is not necessarily in her interest to do so.

5   Risk-neutrality and Standard Debt Contracts

The existence of a unique equilibrium in the risky debt market depends on Proposition 2, which establishes that the optimal debt contract, given proportionate penalties for lying, is the standard debt contract. We apply this result to Proposition 1 in order to establish that p^* is independent of R^* and L^*. This justifies our derivations of the lending supply and demand functions, and hence is critical to the proof of Theorem 9.

Proposition 2 seems to conflict with the results elsewhere in the literature, for example in Boyd and Smith (1994). As we mention in the introduction, the usual finding with stochastic auditing is that an optimal contract leaves some wealth in the borrower's hands, even if the borrower has been caught out in a lie about his total wealth. Boyd and Smith find this result troubling, as standard debt contracts are commonplace, whereas the optimal contracts in their version of the Border and Sobel model are not. So we now discuss the extent to which our analysis can resolve the results that Boyd and Smith draw to our attention.

There are two chief reasons for the difference between their result and ours. First, their

framework requires more than two states. Hence, our results here cannot speak to whether the standard debt contract is indeed optimal. Pursuing that here would take us too far from our chief purpose of analyzing lending with possible bankruptcy when the debt size and interest rates arise as part of a competitive equilibrium. So we defer this first reason to future work.

A second reason that we find the standard debt contract to be optimal in the presence of stochastic auditing, while others do not, is that we work in an environment where agents are risk-averse with CARA utilities, while the tradition in the literature on security design is to work with risk-neutral agents. We now address this difference.

The most common framework, and indeed the one in Gale and Hellwig (1985) and Boyd and Smith (1994), is one with risk-neutral lenders and risk-neutral borrowers. In our setting, with endogenously derived debt sizes, this setting cannot arise:

Proposition 18 *Suppose that a and b are both risk-neutral, so that in each period, $u(c) = c$. Then there can be no equilibrium satisfying inequality (4); i.e., in any equilibrium,*

$$R^*L^* \leq \omega^L.$$

We can nevertheless partially recover the result in Boyd and Smith (1994):

Proposition 19 *Suppose that a is risk-neutral and b has the constant absolute risk-aversion preferences used elsewhere in this paper. Then the standard debt contract is suboptimal.*

Boyd and Smith conjecture that risk-neutrality may contribute to their result, and Proposition 19 supports this conjecture. Moreover, Proposition 2 establishes that the standard debt contract is optimal for any CARA utility where the coefficient of absolute risk-aversion is positive. So a small change in the standard framework, in the form of allowing an infinitesimal amount of risk-aversion, suffices to make the standard debt contract optimal. (This, incidentally, verifies another conjecture of Boyd and Smith, in that they speculate

that the suboptimality of the standard debt contract may not be robust to small changes in the standard model.)

6   Conclusion

This paper characterizes a competitive equilibrium in a debt market where, after a loan contract has been agreed upon, a lender and borrower enter into a costly state verification game. Auditing and lying arise as mixed strategy Nash equilibria in this game, and the competitive equilibrium anticipates this strategic interaction and prices it into the debt markets. Borrowers and lenders can choose to enter a risk-free debt market, in which the issues of default and of costly state verification do not arise. We solve simultaneously for equilibrium in the risk-free debt market, and show that either market can be active or latent. We then use the equilibrium interest rates in both markets to find equilibrium default risk premia, which we decompose into the premia associated with intrinsic risk, i.e., of a borrower having "bad luck," and premia due to strategic risk, i.e., of a borrower fraudulently defaulting.

By making the debt market endogenous, we find several results that differ from those that are standard in the costly state verification literature. The standard debt contract turns out to be optimal, even in the presence of stochastic auditing. Moreover, lenders do not generically have incentive to minimize the deadweight costs of auditing, or even to reduce strategic default. Similarly, for sufficiently high audit costs and accompanying high rates of strategic default, a lender may prefer a costly, perfect audit technology to a costless one.

A Proofs

Proof of Proposition 1. The first-order condition of (7) with respect to p gives

$$u(\omega^H - R^*L^*) = (1 - q^*)u(\omega^H - \omega^L) + q^*u(\omega^H - (1 + \gamma^*)R^*L^*),$$

which says that a must choose q^* in order to make b willing to mix when he receives ω^H. Equation (8) follows from rearranging and substituting (1). Similarly, the first-order condition of (7) with respect to q gives

$$u(\omega^L) = \left(\frac{\pi p^*}{1 - \pi + \pi p^*}\right) u((1 + \gamma^*)R^*L^* - \delta) + \left(\frac{1 - \pi}{1 - \pi + \pi p^*}\right) u(\omega^L - \delta).$$

The first term on the right-hand side is the probability that b's report of $\widehat{\omega}_b^L$ is fraudulent times a's utility of catching b lying. The second term on the right-hand side is the probability that b's reports of $\widehat{\omega}_b^L$ is truthful times a's utility of verifying b's report. And the left-hand side is a's utility of trusting b when b reports $\widehat{\omega}_b^L$. So this says that b must choose p^* in order to make a willing to mix. Rearranging and substituting from (1) yields (9). ∎

Proof of Proposition 2. Using the envelope theorem and differentiating (11) with respect to γ^* gives

$$\frac{\partial EU_a}{\partial \gamma^*} = \pi \frac{\partial p^*}{\partial \gamma}(-u(R^*L^*) + u(\omega^L)).$$

The term in parentheses is negative because $u(\cdot)$ is strictly increasing and because (4) must hold if default is possible. So the effect of γ^* on EU_a depends entirely on the sign of its effect on p^*.

Equation (9) can be rewritten as

$$p^* = \left(\frac{1 - \pi}{\pi}\right) \cdot \left(\frac{u(\omega^L) - u(\omega^L - \delta)}{u((1 + \gamma^*)R^*L^* - \delta) - u(\omega^L)}\right).$$

The denominator increases in γ^*, while the numerator is independent of γ^*. Hence

$$\frac{\partial p^*}{\partial \gamma^*} < 0 \qquad \Rightarrow \qquad \frac{\partial EU_a}{\partial \gamma^*} > 0.$$

∎

Proof of Corollary 3. Plugging γ^* into (8) and (9) gives

$$q^* = \frac{e^{\lambda R^* L^*} - e^{\lambda \omega^L}}{e^{\lambda \omega^H} - e^{\lambda \omega^L}} \tag{31}$$

and

$$p^* = \left(\frac{1 - \pi}{\pi}\right) \cdot \left(\frac{e^{\lambda(\omega^H + \delta)} - e^{\lambda \omega^H}}{e^{\lambda \omega^H} - e^{\lambda(\omega^L + \delta)}}\right), \tag{32}$$

which is easily seen to be strictly decreasing in π. The other results follow from straightforward differentiation. ∎

Proof of Corollary 4. Equation (32) depends only on π and the model parameters $\delta, \lambda, \omega^H$, and ω^L. From Corollary 3, p^* is strictly increasing in π, so we need only solve for the value $\underline{\pi}$ that makes $p^* = 1$. Straightforward algebra yields

$$\underline{\pi} = \left(\frac{e^{\lambda(\omega^H + \delta)} - e^{\lambda \omega^H}}{e^{\lambda \omega^H} - e^{\lambda(\omega^L + \delta)}}\right) / \left[1 + \left(\frac{e^{\lambda(\omega^H + \delta)} - e^{\lambda \omega^H}}{e^{\lambda \omega^H} - e^{\lambda(\omega^L + \delta)}}\right)\right].$$

∎

Proof of Corollary 5. Differentiating q^* with respect to λ gives

$$\frac{\partial q^*}{\partial \lambda} = \frac{(R^* L^* e^{\lambda R^* L^*} - \omega^L e^{\lambda \omega^L})(e^{\lambda(R^* L^*)} - e^{\lambda \omega^L}) - (\omega^H e^{\lambda \omega^H} - \omega^L e^{\lambda \omega^L})(e^{\lambda R^* L^*} - e^{\lambda \omega^L})}{(e^{\lambda \omega^H} - e^{\lambda \omega^L})^2}.$$

The sign of the derivative depends on the numerator, which reduces to

$$(\omega^H e^{\lambda \omega^H} - R^* L^* e^{\lambda(R^* L^*)})(e^{\lambda \omega^L} - e^{\lambda R^* L^*}) < 0.$$

The arguments for the other relationships are similar. ∎

Proof of Proposition 6. The changes with respect to ω^H and π are immediate. Differentiating (12) with respect to R gives

$$\frac{\partial EU_b}{\partial R} = -\frac{\lambda \omega^H - \ln(\pi R)}{\lambda(1 + R)^2} - \frac{1}{\lambda R(1 + R)}$$

$$= -\frac{L^D + \frac{1}{\lambda R}}{1 + R} < 0.$$

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As for λ, observe that (12) can be rewritten as

$$L^D = \frac{\omega^H}{1+R} - \frac{\ln{(\pi R)}}{\lambda(1+R)}.$$

If $R > 1/\pi$, then $\ln{(\pi R)} > 0$ and L^D monotonically increases to $\omega^H/(1+R)$ as λ increases. If $R < 1/\pi$, then L^D monotonically decreases to $\omega^H/(1+R)$ as λ increases. ∎

Proof of Proposition 7. The change with respect to ω_a is immediate. Moreover, L^S depends on ω^H, and ω^L only through p^*. Applying Corollaries 5 and 3 then immediately gives the results with respect to π, ω^H, and ω^L.

Differentiating (16) with respect to R gives

$$\frac{\partial L^S}{\partial R} = \frac{1}{\lambda R(1+R)} - \frac{L^S}{1+R},$$

which is negative whenever $L^S > 1/(\lambda R)$, positive whenever $L^S < 1/(\lambda R)$, and 0 when $L^S = 1/(\lambda R)$. ∎

Proof of Proposition 8. From (21), it is immediate that the last two terms eventually vanish, as their numerators do not depend on R. Since the risky lending supply $L^S(R)$ eventually decreases in R and the risk-free lending supply $L_f^S(R)$ asymptotically approaches $L^S(R)$ from above, it follows that the lending supply in both markets must eventually decrease in R. ∎

Proof of Theorem 9. From equations (12) and (16), we have

$$\frac{\lambda\omega^H - \ln{(R^*\pi)}}{\lambda(R^* + 1)} = \frac{\lambda\omega_a + \ln{[R^*\pi(1 - p^*)]}}{\lambda(R^* + 1)}.$$

Since it is never optimal for a to lend at $R = -1$, we can reduce this to

$$\lambda(\omega^H - \omega_a) - \ln{(R^*\pi)} = \ln{(R^*\pi)} + \ln{(1 - p^*)}$$

$$\Rightarrow \ln R^* = \frac{\lambda(\omega^H - \omega_a) - \ln{(1 - p^*)} - 2\ln \pi}{2}$$

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Exponentiating both sides then gives

$$R^* = \left[\frac{e^{\lambda(\omega^H - \omega_a)}}{\pi^2(1 - p^*)} \right]^{\frac{1}{2}},$$

as desired.

The value of p^* comes directly from (32) in the proof of Corollary 3. Since p^* is fully determined by $(\omega^H, \omega^L, \lambda, \pi, \delta)$, the value of R^* in (23) is likewise unique, given ω_a and these other parameters.

The equilibrium lending demand is calculated from substituting R^* directly into (12) (or into (16)). Since p^*, R^*, and the model parameters uniquely determine L^*, it is also unique, as is R^*L^*. This, in turn, means that q^* and γ^*, found from equation (31) in the proof of Corollary 3 and from Proposition 2, are also uniquely determined. ∎

Proof of Corollary 10. From Corollary 5, p^* increases in ω^L and in δ, hence decreasing the denominator in (23) and thereby increasing R^*.

From Corollary 3, p^* is decreasing in π, so the denominator in (23) increases in π, while the numerator is unaffected. Also note that increasing ω_a decreases the numerator and does not affect the denominator, hence reducing R^*.

Corollary 5 also establishes that p^* increases in λ, hence decreasing the denominator in (23) and increasing R^*. However, the effect of λ on the numerator of R^* depends on the sign of $\omega^H - \omega_a$. If ω_a is sufficiently larger than ω^H, a change in λ will decrease the numerator in R^* enough to offset the decrease in the denominator.

The numerator of R^* increases in ω^H, but p^* decreases in ω^H, so the denominator of R^* also increases in ω^H. ∎

Proof of Theorem 11. The derivation of R_f is analogous to that of R^* in the proof of Theorem 9. Uniqueness follows from the uniqueness of the reservation utilities $\overline{U}_a$ and $\overline{U}_b$. The calculation of L_f then follows from substitution of R_f into (14) or (20). ∎

Proof of Proposition 12. The ratio R^*/R_f reduces to

$$\left[\frac{\pi + (1-\pi)e^{\lambda(\omega^H - \omega^L)}}{\pi^2(1-p^*)}\right]^{\frac{1}{2}}.$$

To see that this ratio exceeds 1, note

$$\pi + (1-\pi)e^{\lambda(\omega^H - \omega^L)} > \pi > \pi^2(1-p^*).$$

As $\pi \nearrow 1$, the left-hand and right-hand sides both approach π. ∎

Proof of Proposition 17. Directly from (23), (25), and (27). ∎

Proof of Proposition 18. In a risk-neutral setting, if p^* and q^* are chosen optimally, then a's individual rationality constraint becomes

$$\omega_a - L^* + \pi(1-p^*)R^*L^* + [1 - \pi(1-p^*)]\omega^L \geq \omega_a$$

$$\Rightarrow (1-\pi)\omega^L + \pi p^*\omega^L \geq (1 - \pi R^*)L^* + \pi p^* R^* L^*. \tag{33}$$

Analogously, b's individual rationality constraint becomes

$$L^* + \pi(\omega^H - R^*L^*) \geq \pi\omega^H + (1-\pi)\omega^L$$

$$\Rightarrow (1 - \pi R^*)L^* \geq (1-\pi)\omega^L. \tag{34}$$

Substituting (34) into (33) gives

$$(1 - \pi R^*)L^* + \pi p^*\omega^L \geq (1 - \pi R^*)L^* + \pi p^* R^* L^*$$

$$\Rightarrow \omega^L \geq R^*L^*,$$

which violates inequality (4). ∎

Proof of Proposition 19. In the case of a risk-neutral lender, p^* becomes

$$p^* = \frac{\delta}{(1+\gamma)R^*L^* - \omega^L - \delta}.$$

If the debt contract is the standard one, then $\gamma = \gamma^* = \omega^H/(R^*L^*) - 1$. In that case, p^* is independent of L^*.

The lender's objective function is to maximize

$$EU_a = \omega_a - [1 - \pi(1 - p^*)R^*]L + [1 - \pi(1 - p^*)]\omega^L.$$

The first-order condition, in the case where $\gamma = \gamma^*$, is

$$\frac{\partial EU_a}{\partial L} = 1 - \pi(1 - p^*)R^* = 0$$

$$\Rightarrow R^* = \frac{1}{1 - p^*}.$$

That is, if the loan size has no effect on p^*, as under the standard debt contract, then the only gross interest rate where there is neither entry nor exit is $1/(1 - p^*)$. We now show that the lender is better off in an equilibrium where R^* is lower than this level, implying γ is not optimally set to a full penalty.

For the general case, differentiating EU_a with respect to the loan size yields

$$\frac{\partial EU_a}{\partial L} = -(1 - \pi(1 - p^*)R^*) - \pi RL^S\frac{\partial p^*}{\partial L} + \pi\frac{\partial p^*}{\partial L}\omega^L = 0$$

$$\Rightarrow 1 - \pi(1 - p^*)R^* = \pi\frac{\partial p^*}{\partial L}(R^*L^S - \omega^L).$$

Since p^* decreases in L, the right-hand side is negative. This implies that, for an optimal contract, $\pi(1 - p^*)R^* < 1$, which is inconsistent with the standard debt contract. ∎

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